

THE GAUSSIAN ISOPERIMETRIC PROBLEM AND THE SELF-SHRINKERS OF MEAN CURVATURE FLOW

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Abstract

We record work done by the author joint with John Ross [27] on stable smooth solutions to the gaussian isoperimetric problem, and we also record work done by the author [26] on two-dimensional self-shrinkers of any co-dimension.

The gaussian isoperimetric problem is related to minimizing a gaussian weighted surface area while preserving an enclosed gaussian weighted volume. The work joint with the author and John Ross [27] studies smooth, stable local minimizers to the gaussian isoperimetric problem. We show that for complete critical solutions of polynomial volume growth, the only stable solutions are the hyper-planes.

As in McGonagle-Ross[27], we also consider the incomplete case of stable solutions to the gaussian isoperimetric problem contained in a ball centered at the origin with their boundary contained in the boundary of the ball. We show that for general dimension, large enough ball, and appropriate euclidean area conditions on the hyper-surface, we get integral decay estimates for the curvature. Furthermore, for the two-dimensional critical solutions in three-dimensional euclidean balls, using a de Giorgi-Moser-Nash type iteration [3, 4, 19] we get point-wise decay estimates for similar conditions. These estimates are decay estimates in the sense that when everything else is held constant, we get zero curvature as the radius of the ball goes to infinity.

We also record work done by the author [26] on the application of gaussian harmonic one-forms to two dimensional self-shrinkers of the mean curvature flow in any co-dimension. Gaussian harmonic one-forms are closed one-forms minimizing a weighted L^2_μ norm in their cohomology class. We are able to show a type of rigidity result that forces the genus of the hyper-surface to be zero if the curvature is smaller than a certain size. This bound for the curvature is larger than that given in the rigidity result of Cao-Li [7] which classifies the self-shrinker as either a sphere or cylinder if the curvature is small enough. We also show that if the self-shrinker satisfies an appropriate curvature condition, then the genus gives a lower bound on the index of the self-shrinking Jacobi operator.

READERS: Professor William P. Minicozzi II (Advisor) and Professor Joel Spruck

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1

Introduction

Many problems in geometric analysis are of a variational nature. These problems often involve finding sub-manifolds or hyper-surfaces inside some ambient space that are local or global extrema of some functional associated with their geometry. Here, we will be investigating particular variational problems related to considering the geometry in euclidean space \mathbb{R}^{n+1} along with a gaussian weight $e^{-\frac{|x|^2}{4}}$.

The usual metric for euclidean space is $g_E = \sum_i (dx^i)^2$, and for every hyper-surface $\Sigma^n \subset \mathbb{R}^{n+1}$, we have a natural area measure $A(\Sigma)$ inherited from \mathbb{R}^{n+1} . Likewise for every open region $\Omega \subset \mathbb{R}^{n+1}$ we have a natural volume measure $V(\Omega)$. We will be considering the gaussian weighted area measure $dA_\mu = e^{-\frac{|x|^2}{4}} dA$ and gaussian weighted volume measure $dV_\mu = e^{-\frac{|x|^2}{4}} dV$.

The traditional isoperimetric problem considers the process of fixing a volume V_0 and finding regions Ω with $V(\Omega) = V_0$ such that Ω minimizes the area of its boundary. Classic results of Schwarz [34] and Steiner [35] show that spheres are the unique minimizers of euclidean space.

One can also consider a local version of an isoperimetric problem. In this case, instead of finding a minimizer Ω_0 and comparing it to all regions Ω with $V(\Omega) = V_0$, we consider finding a minimizer Ω_0 such that it is a minimizer for all regions Ω close to Ω_0 and with $V(\Omega) = V_0$. The boundary of such an Ω_0 is considered a stable solution to the isoperimetric problem. For the euclidean case, Barboas-do Carmo [2] show that the only compact stable solutions are again the spheres.

Here we record work joint between the author and John Ross [27] on the local solutions to the gaussian isoperimetric problem. We consider local minimizers Σ_0 of A_μ that enclose a fixed weighted volume $V_{\mu,0}$ among hyper-surfaces Σ close to Σ_0 and enclosing the same weighted volume. The first derivative test for this variation tells us that $\Sigma_0 = \partial\Omega_0$ must satisfy the mean curvature relation

$H = \frac{1}{2}\langle x, N \rangle + C$ where C is a constant. Borell [5] and Sudakov & Cirel'son [36] show that hyper-planes are global minimizers. We use work done joint with John Ross [27] to show that hyper-planes are in fact the only smooth, two-sided, and complete local minimizers of polynomial volume growth. The fact that hyper-planes are stable is recorded in Theorem 1 of Section 3.

Theorem 1. *Hyper-planes are stable critical hyper-surfaces to the gaussian isoperimetric problem.*

In fact, we show that bounds on the index of the associated Jacobi operator $-L$ for a critical Σ_0 allow us to make statements about Σ_0 splitting off a linear space. Here the index is defined to be the maximal dimension of sub-spaces of weighted volume preserving variations for which $-L$ is negative definite (for example, Σ_0 is stable if and only if the index is 0), and a critical Σ_0 is defined to be satisfying the first derivative test of the gaussian isoperimetric problem. This result is recorded in the following theorem.

Theorem 2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a non-planar critical hyper-surface for the gaussian isoperimetric problem with $\text{Index} = I$ for all compact weighted volume preserving variations. Also, we require $0 \leq I \leq n$. Then*

$$\Sigma = \Sigma_0 \times \mathbb{R}^i, \quad (1.0.1)$$

where $i \geq n + 1 - I$. In particular, there are no non-planar stable solutions.

From Theorem 1 and Theorem 2, we immediately have the following corollary.

Corollary 1. *The only smooth stable solutions of the gaussian isoperimetric problem of polynomial growth are the hyper-planes. There are no solutions of index one.*

We also consider the problem of local solutions $\Sigma \subset B_R(0) \subset \mathbb{R}^{n+1}$ such that $\partial\Sigma \subset \partial B_R(0)$. For any dimension Σ^n , we obtain estimates for $\int_\Sigma |A|^2 dA_\mu$ for certain natural lower bounds on R and upper bounds on the euclidean area of Σ ; see Proposition 2. For the three-dimensional case $\Sigma^2 \subset \mathbb{R}^3$, these integral estimates may be turned into point-wise estimates for $|A|$. McGonagle-Ross [27] uses a Choi-Schoen type argument [8] to establish these point-wise estimates. However, we will use an alternative method based on de Giorgi-Moser-Nash iteration [3, 4, 19]. The point-wise estimates are recorded in the following theorem. We use $\mathcal{O}_C \equiv 1 + |C|$, $\mathcal{O}_R \equiv 1 + R$, and A_E represents the euclidean area of Σ .

Theorem 3. *Let $\Sigma \subset B_R(0) \subset \mathbb{R}^3$ be a smooth stable critical hyper-surface of the gaussian isoperimetric problem with $H = \frac{1}{2}\langle x, N \rangle + C$ and $\partial\Sigma \subset \partial B_R(0)$. Consider any $0 < \theta < 1$. We have*

that there exists a constant D (independent of Σ , θ , and R) such that if $\theta(1 - \theta)R > D\mathcal{O}_C$ and $A_E(\Sigma) < De^{-D\mathcal{O}_C^2}\mathcal{O}_C^n(1 - \theta)^2R^2e^{\frac{(1-\theta)R^2}{16}}$, then

$$\sup_{B_{\theta R}(0)} |A| \leq D\mathcal{O}_C^{\frac{1}{2}}e^{D_0\mathcal{O}_C^2}(1 - \theta)^{-4}A_E(1 + A_E)\mathcal{O}_R^{\frac{1}{4}}e^{-\frac{1}{32}(1-\theta^2)R^2}. \quad (1.0.2)$$

One of the main technical considerations for creating these point-wise estimates is that we must be careful to get estimates that have exponential decay in R for any sub-ball $B_{\theta R}(0)$. Furthermore, we want that it is sufficient that the euclidean area be bounded by exponential growth in R .

Finally we consider the work in McGonagle [26] on two-dimensional self-shrinkers of any co-dimension, $M^2 \subset \mathbb{R}^n$. A self-shrinker of mean curvature flow is a sub-manifold that moves by dilations under the mean curvature flow, and so it must satisfy the mean curvature condition $\vec{H} = \frac{1}{2}\vec{x}^N$, where \vec{x} is the position vector. We use an analog of harmonic one-forms that we call gaussian harmonic one-forms. Gaussian harmonic one-forms are constructed from minimizing the functional $\int_M |\omega|^2 dA_\mu$ over closed one-forms ω in a particular cohomology class. We show that these forms satisfy certain Euler-Lagrange equations that allow us to use them as test functions much like Palmer [31], Ros [33], and Urbano [37] use harmonic one-forms for minimal surfaces.

Our first main result in this direction is a rigidity-type theorem for the genus of M depending on a particular norm of the second fundamental form A . It is expected that bounds on $|A|$ give bounds on the topology of a self-shrinker Σ . An argument pointed out to the author by Professor William Minicozzi shows that if $|A| \leq C|x|^2$ for $C < \frac{1}{8}$, then Σ has finite topology [29]. For any two-dimensional self-shrinker Σ , we have that $|H|^2 \leq 2|A|^2$. For a self-shrinker, we then get that $|x^N|^2 \leq 8C|x|^2$. So, we have that $|x^T|^2 \geq (1 - 8C)|x|^2 > 0$ for $|x|$ sufficiently large. So, from Morse Theory, we have that the topology of $\Sigma \cap B_R(0)$ is constant for large enough R .

Our first main result is the following.

Theorem 4. *If $M^2 \subset \mathbb{R}^N$ is an orientable self-shrinker of polynomial volume growth with genus $g \geq 1$, then*

$$\sup_{x \in M, v \in T_x M, |v|=1} A^\alpha(v, i)A^\alpha(i, v) \geq \frac{1}{2}. \quad (1.0.3)$$

This theorem should be compared with a rigidity theorem of Cao-Li [7]. Cao-Li show that if $|A| \leq \frac{1}{2}$ (here we have renormalized their result to match our definition of self-shrinker) then M must be a sphere or cylinder. Our result shows that for $M^2 \subset \mathbb{R}^3$, we have that if the maximum of the absolute value of the principal curvatures $|\kappa_i| \leq \frac{1}{2}$, then the genus must be zero.

Our second large result in this direction is a result on how the genus of M gives us lower bounds

on the index of the appropriate Jacobi operator L if M satisfies a certain curvature condition.

Theorem 5. *Let $M^2 \subset \mathbb{R}^n$ be an orientable self-shrinker of polynomial volume growth. If*

$\sup_{p \in M} \inf_{\{\eta_\alpha\} \text{ orth.}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| \leq \delta < 1$, then the index of L acting on scalar functions on M has a lower bound given by

$$\text{Index}_M(L) \geq \frac{g}{n}. \quad (1.0.4)$$

1.1 Notation

Here we give a consolidated list of notation that will be used throughout our arguments.

Generally we use M^m to be a sub-manifold of N^n of any co-dimension. We denote hyper-surfaces of N^n by Σ .

The covariant derivative on M or Σ is denoted by ∇ . The covariant derivative on N is denoted by $\bar{\nabla}$. So, $\nabla v = (\bar{\nabla} v)^T$. The covariant derivative on the normal bundle of M is denoted by ∇^N , so that $\nabla^N v = (\bar{\nabla} v)^N$.

We will often need to make use of the second covariant derivative. We will use the notation ∇^2 for the second covariant derivative for the ∇ connection on the sub-manifold M , $\bar{\nabla}^2$ for the second covariant derivative connection on the ambient manifold N , and $\nabla^{N,2}$ for the second covariant derivative of the normal connection ∇^N . For example, $\nabla_{X,Y}^{N,2} V = \nabla_X^N(\nabla_Y^N V) - \nabla_{\nabla_X Y}^N V$.

For the general co-dimensional case of $M^m \subset N^n$, we will be using $A_{ij} = -\nabla_i^N \partial_j$. Let η_α be a frame for the normal bundle. We will often need to use the notation $A_{ij}^\alpha = -\langle \nabla_i^N \partial_j, \eta_\alpha \rangle$.

For the co-dimension one case $M^m \subset N^{m+1}$, we make use of the standard convention $A_{ij} = -\langle \bar{\nabla}_i N, \partial_j \rangle$.

For the case of $N^n = \mathbb{R}^n$, we will be needing to make use of certain second order elliptic operators. Let x be the position vector in Euclidean space. For scalar functions $u \in C^2(M^m)$, we define

$$\mathcal{L}u = \Delta u - \frac{1}{2} \nabla_{x^T} u \quad (1.1.1)$$

$$L u = \mathcal{L}u + |A|^2 u + \frac{1}{2} u. \quad (1.1.2)$$

For any C^1 Euclidean vector field V defined along M^m , we define $\nabla V = (\bar{\nabla} V)^T$. We are interested in the euclidean case where we denote the ambient euclidean connection by ∇^E . If V is

C^2 , we define:

$$\mathcal{L}^M V = \nabla_{ii}^2 V - \frac{1}{2} \nabla_{x^T} V, \quad (1.1.3)$$

$$\mathcal{L}^E V = \nabla_{ii}^{E,2} V - \frac{1}{2} \nabla_{x^T}^E V, \quad (1.1.4)$$

$$\mathcal{L}^E V = \mathcal{L}^E V + |A|^2 V + \frac{1}{2} V, \quad (1.1.5)$$

$$(1.1.6)$$

As explained in Section 2.3, all variations are considered to be normal.

While computing estimates for stable solutions to the gaussian isoperimetric problem in $B_R(0)$, we will often be making use of arbitrary constants. We will often need to increase the size of constants so that we may use results from other lemmas or propositions. To accomplish this, we write every statement in such a way that they remain valid if we increase the size of the arbitrary constants.

While investigating gaussian harmonic forms, we will often need to make use of euclidean vectors and their euclidean components. We will use indices such as a, b, \dots to denote euclidean vectors or components. For example, the vectors $\{\partial_a\}$ are an orthonormal basis in \mathbb{R}^n , and for any tangential vector v we may write $v = v^a \partial_a$.

2

Background

2.1 The f -Area Functional and f -Minimal Surfaces

Consider a manifold N^n and a sub-manifold $M^m \subset N^n$. In geometric analysis, natural quantities to consider are the m -dimensional areas of compact subsets of M . Very important questions are related to how these m -dimensional areas vary as we bend or change M on compact sets. Using a parametric approach, we model these variations by considering smooth functions $F(x, t) : M^m \times \mathbb{R} \rightarrow N^n$ such that $F(x, 0) = x$, and for each compact time interval $[t_0, t_1]$, the set $\{x \in M^m | F(x, t) \neq x \text{ and } t_0 \leq t \leq t_1\}$ is contained in a compact set. Also, for clarity of notation, we use $U_t \equiv F(U, t)$, and for coordinates x_i on M^m we use $F_i \equiv \frac{\partial F}{\partial x_i}$ and $F_t \equiv \frac{\partial F}{\partial t}$.

We will also need to consider both weighted areas and weighted volumes. For now, we will only consider weighted areas, and we will hold our discussion of weighted volumes until we review the Gaussian Isoperimetric Problem.

Definition 1. For a smooth function $f : M^m \rightarrow \mathbb{R}$, we define the weighted f -area of a compact subset $U \subset M$ by

$$A_f(U) = \int_U e^f dA. \quad (2.1.1)$$

One should note that by a conformal change of metric, it is possible to consider the weighted f -area to be the regular area functional under a different metric. However, when studying the Gaussian Isoperimetric Problem, we will have need to consider the area functional in the form given by (2.1.1).

We wish to consider manifolds M that are minimizers or local minimizers for the f -area functional. Consider a compact variation $F(x, t)$ and some compact domain $U \subset M$ such that on some

compact time interval I containing 0, we have that $F(x, t) = x$ on $(M \setminus U) \times I$. Then, we have that $\frac{d}{dt}A_f(U_t)|_{t=0} = 0$ and $\frac{d^2}{dt^2}A_f(U_t)|_{t=0} \geq 0$. Therefore, to consider the local minimizers of A_f we must consider the first and second variations of A_f .

Since the f -area functional is invariant under diffeomorphic re-parameterization of M , it is often convenient to consider variations $F(x, t)$ such that $\frac{\partial}{\partial t}F(x, t)$ is normal to the tangent space of $TF(\cdot, t)$. Moving from a general compact variation to a compact normal variation is accomplished by solving the appropriate ODE that arises from re-parameterizing by diffeomorphisms.

By considering M as a domain of parameterization for every time slice $F(\cdot, t)$, we have that

$$A_f(U_t) = \int_U e^{f \circ F} dA_t. \quad (2.1.2)$$

Here dA_t is the euclidean area element of each time slice. A standard computation [9] gives that $\frac{d}{dt}dA_t|_{t=0} = \langle H, \frac{\partial F}{\partial t} \rangle|_{t=0}$. So we clearly have

$$\frac{\partial}{\partial t}A_f(U_t)\Big|_{t=0} = \int_U e^f \langle \bar{\nabla} f + H, F_t \rangle dV. \quad (2.1.3)$$

We see that a necessary condition for M to be a local minimizer for A_f is that $\bar{\nabla} f + H \equiv 0$.

Definition 2. A sub-manifold $M \subset N$ is called ***f -minimal*** if $\bar{\nabla} f + H \equiv 0$.

The special case of $f \equiv 0$ gives the regular area functional, and it is standard to call these sub-manifolds minimal.

Now, a standard computation (see proof of Lemma 2 given in the appendix) gives the second variation of the f -area for an f -minimal sub-manifold.

Lemma 1. Let $M^m \subset N^n$ be an f -minimal sub-manifold with compact normal variation $F(x, t)$. Let U be a compact domain of M and $U_t = F(U, t)$. We have that

$$\begin{aligned} \frac{\partial^2}{\partial t^2}A_f(U_t)\Big|_{t=0} &= - \int_U e^f \left(\left\langle \nabla_{ii}^{N,2} F_t + \langle A_{ij}, F_t \rangle A_{ij}, F_t \right\rangle + \bar{\text{Ric}}(F_t, F_t) \right) dA \\ &\quad - \int_U e^f \bar{\text{Hess}}_f(F_t, F_t) dA. \end{aligned} \quad (2.1.4)$$

When the co-dimension is one and M is two sided, we use scalar functions to represent variations

$F(x, t)$. We use $u \equiv \langle F_t, N \rangle$ and then have that (2.1.4) becomes

$$\left. \frac{\partial^2}{\partial t^2} A_f(U_t) \right|_{t=0} = - \int_U e^f u (\Delta u + |A|^2 u + \overline{\text{Ric}}(N, N)u + \overline{\text{Hess}}_f(N, N)u) dA. \quad (2.1.5)$$

For the case of unweighted euclidean area, we use $f \equiv 0$, and get

$$\left. \frac{\partial^2}{\partial t^2} A(U_t) \right|_{t=0} = - \int_U u \Delta u + |A|^2 u^2 + \overline{\text{Ric}}(N, N)u^2 dV. \quad (2.1.6)$$

Therefore, a minimal hyper-surface is stable if and only if the operator $-(\Delta + |A|^2 + \overline{\text{Ric}}(N, N))$ is positive.

The property of a hyper-surface Σ being minimal and stable is strong. It puts restrictions on Σ in many ways including topology and singularities. For example, the only stable complete minimal hyper-surfaces in \mathbb{R}^3 are the planes [12, 18, 32, 33].

2.2 Isoperimetric Problems

For a region $\Omega \subset N$, the volume of Ω is defined by the n -dimensional volume in N , $\int_{\Omega} dV_N$. If $\partial\Omega$ is C^1 , then the area of Ω is defined by the area element on $\partial\Omega$.

Isoperimetric problems are concerned with minimizing the area of the boundary of a region in space over all regions with a fixed volume V . More specifically, two questions of importance in isoperimetric problems are the following:

1. Given a fixed volume V , what is the infimum of boundary area over all regions of volume V in a manifold M ?
2. For a fixed volume V , is the infimum realized by an actual minimizer? In other words, does there exist a region Ω whose boundary $\partial\Omega$ has area realizing the infimum?

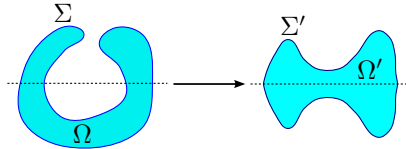


Figure 2.1: A Steiner Symmetrization

For the Euclidean case of $N = \mathbb{R}^{n+1}$, a classic result of Schwarz [34] and Steiner [35] show that balls are the only minimizers. The proof uses a rearrangement technique called Steiner Sym-

metrization to show that any minimizer must be spherically symmetric. A Steiner Symmetrization is a vertical rearrangement over a hyperplane that preserves volume, reduces area, and makes the domain symmetric about the hyperplane. See Figure 2.1 above.

A local minimizer for an isoperimetric problem is a region Ω of fixed volume V that has minimal boundary area amongst all regions that are close to Ω . Another important question for isoperimetric problems is the following:

3. Do there exist local minimizers that aren't global minimizers?

An important tool for studying solutions to these three questions uses functions to parameterize compact variations that preserve the enclosed volume. Consider $\Sigma = \partial\Omega$ (so Σ is two-sided) that is at least C^2 . For a given variation $F(x, t)$, we again use a scalar function $u \equiv \langle F_t, N \rangle$.

Instead of dealing directly with the volume of the enclosed region $\Omega_t \equiv F(\Omega, t)$, we deal with the signed difference in volume. See Figure 2.2 below. If ω is the volume form of N , then the signed volume change is given by

$$V(t) = \int_{[0, t] \times \Sigma} F^* \omega. \quad (2.2.1)$$

Note, that since $\{(x, s) | \{F(x, s) \neq x \text{ and } s \in [0, t]\} \text{ is compact, we have that (2.2.1) is well-defined.}$

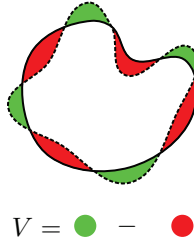


Figure 2.2: The V functional

Barbosa, do Carmo, and Eschenburg [1] show that

$$\left. \frac{\partial}{\partial t} V(t) \right|_{t=0} = \int_{\Sigma} u dA. \quad (2.2.2)$$

Therefore, variations u that preserve volume are precisely represented by those $u \in C_0^1(\Sigma)$ such that $\int_{\Sigma} u dA = 0$.

Hence, for a hyper-surface Σ to be a local minimum among hyper-surfaces enclosing the same volume, it is necessary that $\int_{\Sigma} H u dA = 0$ for all $u \in C_0^1(\Sigma)$ satisfying $\int_{\Sigma} u dA = 0$. This represents a first derivative test for the area functional under the constraint of using hyper-surfaces enclosing a fixed volume, and it implies that the mean curvature H is constant on Σ .

Barbosa, do Carmo, and Eschenburg [1] show that for a critical hyper-surface Σ (a hyper-surface of constant mean curvature), one has that the second variation of the area functional is also given by (2.1.6). In earlier work on the Euclidean case, Barbosa and do Carmo [2] use the generators of homotheties $u = \langle x, N \rangle$ and constant functions to show that spheres are the only compact, stable local minima of the isoperimetric problem in Euclidean space. Barbosa, do Carmo, and Eschenburg [1] later generalize this to show that if the ambient manifold is simply connected and of constant curvature, then the only compact, stable local minima are geodesic spheres.

2.3 Mean Curvature Flow and Self-Shrinkers

A smooth family of sub-manifolds $M_t \subset N$ parameterized by $F(x, t) : M_0 \times \mathbb{R} \rightarrow N$ is said to be moving by the mean curvature flow if

$$\left[\frac{\partial \vec{F}}{\partial t}(x, t) \right]^N = -\vec{H}. \quad (2.3.1)$$

Note that geometrically we can only specify the normal part of the variation; we will have the same family of sub-manifolds if we change each M_t by a diffeomorphism of itself. That is, geometrically, we can not specify the tangential part of the variation. This will receive more explanation when we introduce self-shrinkers of the mean curvature flow.

The definition of the mean curvature flow that we have introduced is a parametric model. It is assumed throughout the existence of the flow that the topology is unchanging. Non-parametric models of geometric flow allow for an analysis of the mean curvature flow where the topology is changing. These models include level-set mean curvature flow [14] and Brakke flow [6]. It should be noted that these approaches are not equivalent.[22] The price we pay for using a parametric flow is that as M_t approaches a change in topology, the solution will become non-smooth as its curvature blows up.

For a compact hyper-surface, the non-existence of the flow for all time is guaranteed by the maximum principle and the fact that a sphere of radius R will collapse by homothety to its center in time $T = \frac{R^2}{2n}$. Therefore, for a random initial sub-manifold, it is expected that a singularity will occur during the mean curvature flow.

To understand the mean curvature flow for non-parametric models, we need to understand the behavior of the parametric model as M_t approaches the singularity. For convenience, we consider the time of the flow to be defined $[-T, 0)$ with a singularity at $t = 0$. An important type of sub-manifold

for understanding this behavior are the self-shrinkers of mean curvature flow.

Definition 3. A sub-manifold $M \subset N$ is a **self-shrinker** of the mean curvature flow if

$$\vec{H} = \frac{\vec{x}^N}{2}. \quad (2.3.2)$$

A self-shrinker M flows under mean curvature flow by dilations to the origin; so, if we choose $M_{-1} = M$, then $M_t = \sqrt{-t}M$. Note, that the derivatives of this parameterization of the flow are not orthogonal to the tangent space of M_t , and so it is not true that $\partial F/\partial t$ is the mean curvature vector \vec{H} . However, it is true that this evolution of a self-shrinker satisfies the mean curvature flow (2.3.1).

A singularity of the mean curvature flow is defined to be of **Type I** if

$$\sup_{M_t} |A|^2 \leq \frac{C}{-t}. \quad (2.3.3)$$

In the case of a flow of compact hyper-surfaces Σ_t , Huisken [20] shows that if a singularity is of Type I, then the flow is asymptotic to a self-shrinker as it approaches the singular time.

Ilmanen [22] and White [38] show that in general, M_t weakly converges as varifolds to a self-shrinker in the sense of Brakke mean curvature flow. Around the singularity $(x_0, 0)$, one performs parabolic blow-up $M_t^\lambda = \lambda(M_{\lambda^{-2}t} - x_0)$. Ilmanen[21] and White[38] both show that for every sequence $\lambda_i \rightarrow 0$, there is a subsequence λ_{i_j} such that $M_t^{\lambda_{i_j}}$ converges weakly as varifold flows to an integral Brakke flow. Note that since $M_t^{\lambda_{i_j}}$ is a parabolic blow-up, we have that any limit is invariant under parabolic dilations. So the limit must represent a self-shrinking Brakke flow [22].

Recent progress on the uniqueness of the tangent flows arising from these parabolic blow-ups have been made by Colding-Minicozzi [11]. They show that if the first singular time has a blow up that is a multiplicity one cylinder, then the limits of the many different sequences of parabolic blow-ups must be unique. That is, all other converging parabolic blow-ups must converge to the same multiplicity one cylinder.

Let

$$\rho(x, t) = [4\pi(t - t_0)]^{-m/2} e^{-|x - x_0|^2/4(t - t_0)} \quad (2.3.4)$$

be the backwards m -dimensional heat kernel. Huisken [20] shows that we have a monotonicity formula

$$\frac{d}{dt} \int_{M_t} \rho(x, t) dA = - \int_{M_t} \rho(x, t) \left| \vec{H} - \frac{(\vec{x} - \vec{x}_0)^N}{2(t - t_0)} \right|^2 dA. \quad (2.3.5)$$

This monotonicity formula is an important tool used by Huisken [20], Ilmanen [22], and White [38] to establish the above results on the convergence of mean curvature flows to self-shrinkers.

The m -dimensional backwards heat kernel is also related to self-shrinkers by the fact that self-shrinkers are minimal submanifolds of \mathbb{R}^n for the conformal gaussian metric $ds^2 = e^{-|x|^2/2m} \sum_i dx_i^2$ [10]. Thus, we see that self-shrinkers are related to the geometry of conformal changing the euclidean metric by a gaussian function.

Variations for self-shrinkers are often represented by functions defined using the euclidean metric $u \equiv \langle F, N_E \rangle_E$. Using the appropriate $f = e^{-\frac{|x|^2}{4}}$ in (2.1.4), one obtains that

$$\left. \frac{\partial^2}{\partial t^2} A_f(U_t) \right|_{t=0} = - \int_U e^{-|x|^2/4} u L u dA, \quad (2.3.6)$$

where $L u = \Delta u - \frac{1}{2} \nabla_{x^T} u + |A|^2 u + \frac{1}{2} u$. Colding-Minicozzi [10] analyze the operator L and its eigenfunctions to show that the only generic self-shrinkers are spheres and cylinders $S^{n-k} \times \mathbb{R}^k$.

2.4 The Gaussian Isoperimetric Problem

Consider Euclidean space \mathbb{R}^{n+1} with a gaussian conformal metric $ds^2 = e^{-|x|^2/2(n+1)} \sum_i dx_i^2$. We have a weighted volume form on \mathbb{R}^{n+1} given by $\omega_\mu = e^{-|x|^2/4} \omega_E$ where ω_E is the standard euclidean volume form for \mathbb{R}^{n+1} . We let μ be the weighted volume measure defined by $d\mu \equiv e^{-|x|^2/4} d\mathcal{L}^{n+1}$. We also let A_μ be the weighted area measure defined by $dA_\mu \equiv e^{-|x|^2/4} dH^n$ where H^n is the n -dimensional Hausdorff measure on \mathbb{R}^{n+1} . Note that since both μ and A_μ are defined using the same weight, they can't both be simultaneously identified as being the area and volume measures for a single conformal change of the euclidean metric.

Now, consider a region $\Omega \subset \mathbb{R}^{n+1}$, and let Ω_ϵ be the ϵ -neighborhood of Ω . Fix a weighted volume D . The Gaussian Isoperimetric Problem considers finding regions Ω of fixed weighted volume $\mu(\Omega) = D$ such that the ϵ -neighborhoods Ω_ϵ all have minimal weighted volume. That is, for any other region Ω' with $\mu(\Omega') = D$, one has $\mu(\Omega_\epsilon) \leq \mu(\Omega'_\epsilon)$ for small enough ϵ .

Borell [5] and Sudakov & Cirel'son [36] show that the absolute minimizers of the gaussian isoperimetric problem are half-spaces. That is, for $0 < D \leq \mu(\mathbb{R}^{n+1})$ and any given half-space S with $\mu(S) = D$, we have $\mu(S_\epsilon) \leq \mu(\Omega_\epsilon)$ for any Ω with $\mu(\Omega) = D$.

Just like the Euclidean case, we may consider a local form of the gaussian isoperimetric problem. We consider regions Ω with smooth boundary $\Sigma = \partial\Omega$, and Ω minimizing the weighted volume of its ϵ -neighborhoods when compared with regions Ω' close to Ω . First, from the formula for the first

variation of the weighted volume (2.4.2), we have that $\mu(\Omega_\epsilon) = \mu(\Omega) + A_\mu(\Sigma)\epsilon + \mathcal{O}_\Sigma(\epsilon^2)$.

Therefore, we see that if we compare two regions Ω and Ω' ($\Sigma' = \partial\Omega'$) of the same weighted volume, then in order for Ω to have ϵ -neighborhoods of smaller weighted volume, it is necessary that $A_\mu(\Sigma) \leq A_\mu(\Sigma')$. Hence, if Ω is a local minimizer of the gaussian isoperimetric problem, then it is necessary that $A_\mu(\Sigma) \leq A_\mu(F_t(\Sigma))$ for all compact, weighted volume preserving variations $F(t, x)$. Thus, we are lead to studying the problem of finding local minimizers Σ for $A_\mu(\Sigma)$ and compact variations preserving weighted volume.

For determining those functions representing variations preserving weighted volume, we consider a weighted volume functional similar to (2.2.1).

Definition 4. For a given compact variation $F(x, t)$, the gaussian weighted volume functional V_μ is defined by

$$V_\mu(t) = \int_{[0, t] \times \Sigma} F^* \omega_\mu. \quad (2.4.1)$$

In the case that $\Sigma = \partial\Omega$, we have that for small $|t|$, $V_u(t)$ measures the change in the weighted volume from $\mu(\Omega) = \mu(F_0(\Omega))$ to $\mu(F_t(\Omega))$. Much like the Euclidean case [2], one has

$$\left. \frac{\partial}{\partial t} V_\mu(t) \right|_{t=0} = \int_\Sigma u dA_\mu. \quad (2.4.2)$$

From (2.1.3) we see that the first variation of A_μ is given by

$$\left. \frac{\partial}{\partial t} A_\mu(t) \right|_{t=0} = \int_\Sigma \left(H - \frac{\langle x, N \rangle}{2} \right) u dA_\mu. \quad (2.4.3)$$

Therefore, we seek hyper-surfaces such that $\int_\Sigma (H - \langle x, N \rangle / 2) u dA_\mu = 0$ for all $u \in C_0^1(\Sigma)$ such that $\int_\Sigma u dA_\mu = 0$. This occurs exactly when $H - \langle x, N \rangle / 2$ is a constant function on Σ . So, we are lead to the following definition.

Definition 5. A hyper-surface Σ is a **critical hyper-surface to the Gaussian Isoperimetric Problem** if and only if

$$H = \frac{\langle x, N \rangle}{2} + C, \quad (2.4.4)$$

where C is a constant.

For the second variation of A_μ at a critical hyper-surface of the Gaussian Isoperimetric Problem, we will make use of the operators $\mathcal{L}u = \Delta u - (1/2)\nabla_{x^T} u$ and $\mathcal{L}u = \mathcal{L}u + |A|^2 u + (1/2)u$ acting on functions $u \in C_0^2(\Sigma)$. As discussed before, these operators are also of importance in the case of

the self-shrinkers of the mean curvature flow. The operator L gives the second variation of A_μ for a self-shrinker [10]. In fact, we also have that if Σ is a critical point to the Gaussian Isoperimetric Problem, then the operator L gives the second variation of A_μ for variations preserving the weighted volume. We represent compact normal variations $F(x, t)$ of Σ by functions $u \equiv \langle F_t, N \rangle$. Much like the euclidean case [2], we may use (2.4.2) to show that u represents a weighted volume preserving compact normal variation if and only if $\int_\Sigma u dA_\mu = 0$.

Lemma 2. *If Σ is a critical hyper-surface to the Gaussian Isoperimetric Problem and let $F(x, t)$ be a compact volume preserving normal variation. Let $\text{suppt } F_t \subset U \subset \Sigma$, $U_t = F(U, t)$, and $u = \langle F_t, N \rangle$. Note that $\int_\Sigma u dA_\mu = 0$. We have that*

$$\left. \frac{\partial^2}{\partial t^2} A_\mu(U_t) \right|_{t=0} = - \int_U u L u dA_\mu. \quad (2.4.5)$$

Therefore, we are lead to consider the stability of the operator L for the space of functions $u \in C_0^2(\Sigma)$ such that $\int_\Sigma u dA_\mu = 0$.

Definition 6. *A critical hyper-surface Σ of the Gaussian Isoperimetric Problem is said to be **stable** if*

$$- \int_\Sigma u L u dA_\mu \geq 0 \quad (2.4.6)$$

for all $u \in C_0^2(\Sigma)$ such that $\int_\Sigma u dA_\mu = 0$.

Furthermore for critical hyper-surfaces Σ , we will also be concerned with the consequences of bounds on the maximal dimension of sub-spaces of weighted volume preserving variations u where $-L$ is negative definite.

Definition 7. *For a critical hyper-surface Σ of the Gaussian Isoperimetric Problem and a compact, connected domain $\Omega \subset \Sigma$, the **Index $_U L$** is defined to be the maximal dimension of subspaces V contained in the space $\{u \in C_0^2(U) \mid \int_\Sigma u dA_\mu = 0\}$ such that $Q(u) \equiv - \int_\Sigma u L u dA_\mu$ is negative definite.*

For any increasing exhaustion Ω_i of Σ by compact, connected domains, we define

$$\mathbf{Index}_\Sigma L = \sup_{U_i} \mathbf{Index}_{U_i} L. \quad (2.4.7)$$

Remark 1. *Note, that the sequence $\mathbf{Index}_{U_i} L$ is increasing since $U_i \subset U_{i+1}$.*

We will also be concerned with curvature decay estimates for $\Sigma^n \subset B_R(0) \subset \mathbb{R}^n$ with $\partial\Sigma \subset$

$\partial B_R(0)$. Curvature estimates have many uses. For example, they allow one to estimate the size of domain for which a hyper-surface is a graph over a particular tangent plane, and they also measure in a certain sense how far a hyper-surface is from planar [9]. By a decay estimate, we mean that if we fix the size of the constant C in the curvature condition $H = \frac{1}{2}\langle x, N \rangle$ and fix a reasonable area growth condition, then as $R \rightarrow \infty$ our estimates give us that the curvature goes to $|A| = 0$.

In Proposition 2, we show that for large enough θR (here, $0 < \theta < 1$) and for smooth, stable critical points satisfying certain area growth bounds, that we get decay estimates for $\int_{B_R(0) \cap \Sigma} |A|^2 dA_\mu$. The starting point for the derivation of these estimates is the stability inequality $-\int_\Sigma u \mathcal{L} u dA_\mu \leq 0$ for $\int_\Sigma u dA_\mu = 0$. Then using a De Giorgi-Moser-Nash iteration type argument, we show in Theorem 3 that for the two dimensional case of $\Sigma^2 \subset B_R(0) \subset \mathbb{R}^3$, we get point-wise decay estimates for $|A|$ on sub-balls $B_{\theta R}(0)$.

2.5 Harmonic One Forms

A harmonic one-form ω is a closed one-form ($d\omega = 0$) on a manifold M such that ω satisfies the Euler-Lagrange equations for minimizing the norm $\int_M |\omega|^2 dV$. In particular is the case of constructing harmonic one-forms Ω on a two-dimensional manifold M^2 .

Let τ_i be an orthonormal frame of one-forms on M . We have that $d = \tau_i \wedge \nabla_i$ [24]. Therefore, we see that the condition of ω being closed is equivalent to $\nabla \omega$ being symmetric. If ω_0 minimizes $\int_M |\omega|^2 dA$ in its cohomology class, then we see that it necessary that $\int_M \langle \omega, df \rangle dA = 0$ for all $f \in C_0^\infty(\Sigma)$. Therefore, we have that $\delta \omega = 0$. However, for one-forms we have that $\delta = -\text{Div}$, and therefore we have that it is necessary that $\text{Tr } \nabla \omega = 0$. So, it is standard to make the following definition.

Definition 8. For a manifold M , a one-form ω on M is called a **harmonic one-form** if

1. ω is closed ($\nabla \omega$ is symmetric).
2. ω is co-closed ($\text{Tr } \nabla \omega = 0$).

For an orientable two-dimensional manifold M^2 , the **genus** is defined to be the maximal number of embedded Jordan curves that don't separate M into two or more open domains. The genus g of M gives lower bounds on the dimension of the subspace of harmonic one-forms [15, 23]. Using any non-trivial Jordan curve γ that doesn't separate M , we may construct two linearly independent harmonic-one forms. First consider a neighborhood N of γ such that γ separates N into two components N_1 and N_2 . We define a function $f \in C^\infty(M \setminus \gamma)$ such that $f \equiv 0$ on $M \setminus N_1$, and

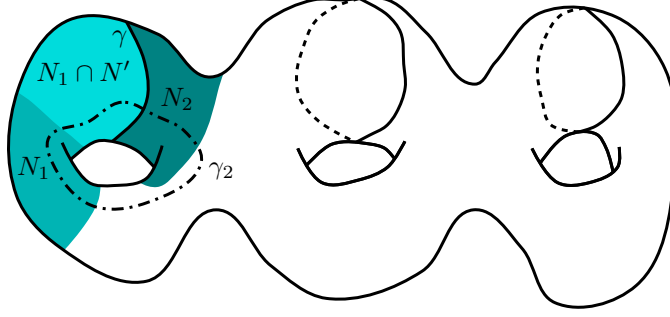


Figure 2.3: Set up of the Jordan curves

such that there is a neighborhood N' of γ such that $f \equiv 1$ on $N_1 \cap N'$. See Figure 2.3 above. We now define the one-form $\omega_1 \equiv df$, and we note that ω_1 is $C_0^\infty(M)$. Also, note that since γ does not separate M , there exists another Jordan curve γ_2 such that $\int_{\gamma_2} \omega_1 \neq 0$.

Using the Hodge star operator $*$, we can then construct the one-form $\omega_2 = *\omega_1$. Note that for a Jordan curve γ_3 close to γ , we have that $\int_{\gamma_3} \omega_2 \neq 0$ while $\int_{\gamma_3} \omega_1 = 0$. Therefore, ω_1 and ω_2 are in different cohomology classes. So, by minimizing the norm $\int_M |\omega|^2 dV$ in their respective cohomology classes, we obtain two linearly independent harmonic one-forms. Furthermore, the above construction works for g disjoint non-separating Jordan curves, and we see that if M has genus g then we may construct at least $2g$ linearly independent L^2 harmonic one-forms.

Harmonic one-forms have important applications, including their use to create lower estimates for the index of geometric operators. Palmer [31] uses harmonic one-forms to show that for compact constant mean curvature surfaces $\Sigma \subset \mathbb{R}^3$, one has that the operator associated to the second variation of the energy of the gauss map $N : \Sigma \rightarrow S^2$ ($E(\Sigma) = \int_\Sigma |dN|^2 dA$) has index bounded below by $\lfloor 1 + g/2 \rfloor$. Ros [33] uses harmonic one-forms to show for two-dimensional minimal surfaces Σ (regardless of orientability) in quotients of \mathbb{R}^3 , one has a lower bound for the index of the Jacobi operator depending on the genus of either Σ or the orientable double cover of Σ . This and other tools show that there are no stable two-sided minimal surfaces in \mathbb{R}^3 . Urbano [37] uses harmonic one-forms to study the index of minimal surfaces immersed in quotients of S^3 and $S^2 \times \mathbb{R}$.

For studying two-dimensional self-shrinkers $M^2 \subset \mathbb{R}^n$, instead of using harmonic one-forms, we will make use of closed forms that minimize the weighted norm $\int_M |\omega|^2 dA_\mu$ in their cohomology class. Here we record the author's work from [26] on such forms. Closed one-forms satisfying the respective Euler-Lagrange equation will be called gaussian-harmonic one-forms. The derivation of the Euler-Lagrange equation is similar to the euclidean case above, and we make the following definition.

Definition 9. For a manifold $M \subset \mathbb{R}^n$, a one-form on M is called a **gaussian-harmonic one-form** if

1. ω is closed ($\nabla\omega$ is symmetric),
2. ω satisfies $\text{Tr } \nabla\omega = \frac{1}{2}\omega(x^T)$.

Just like the euclidean case, we consider the effect of the associated Jacobi operator L on the euclidean coordinate functions of gaussian harmonic one-forms. This will allow us to link a curvature bound to the genus of the self-shrinker, and we will also link the genus of a two-dimensional self-shrinker to lower bounds on the index of L for specific curvature bounds.

3

Gaussian Isoperimetric Problem

Here we consider the work joint between the author and John Ross [27] on the gaussian isoperimetric problem. We consider smooth orientable hyper-surfaces $\Sigma \subset \mathbb{R}^{n+1}$ such that Σ satisfies polynomial volume growth.

3.1 Stability of Planar Hyper-surfaces

Theorem 1. *Hyper-planes are stable critical hyper-surfaces to the gaussian isoperimetric problem.*

Proof. First, note that by rotation we may consider the plane $\{x_{n+1} = D\}$. If we let $x = (x', x_{n+1})$, then on the plane $\{x_{n+1} = D\}$, we have that $e^{-\frac{|x|^2}{4}} = e^{-\frac{D^2}{4}} e^{-\frac{|x'|^2}{4}}$. Therefore, the stability of $\{x_{n+1} = D\}$ is entirely equivalent to the stability of $\{x_{n+1} = 0\}$.

Following Kapouleas-Kleene-Møller [25], we compare the second order operator L to the harmonic oscillator. We have that

$$L u = e^{\frac{|x|^2}{8}} \left(\Delta - \frac{|x|^2}{16} + \frac{1}{4}(n+2) \right) \left(e^{-\frac{|x|^2}{8}} u \right). \quad (3.1.1)$$

The operator

$$H = \Delta - \frac{|x|^2}{16} + \frac{1}{4}(n+2), \quad (3.1.2)$$

may be viewed as a shifted version of the harmonic oscillator operator $\overline{H} \equiv \Delta - |x|^2$. Using a change

of variables $x = 2y$, we have that

$$H = \frac{1}{4}\Delta_y - \frac{|y|^2}{4} + \frac{1}{4}(n+2), \quad (3.1.3)$$

$$= \frac{1}{4}\overline{H}_y + \frac{1}{4}(n+2). \quad (3.1.4)$$

The eigenvalues of \overline{H}_y are well-known to be $n + 2k$ for integers $k \geq 0$, and their eigenfunctions are formed by the product of Hermite polynomials with $e^{-\frac{|y|^2}{2}}$. Therefore, the eigenvalues of H are

$$-\lambda = \frac{n + 2k - n - 2}{4}, \quad (3.1.5)$$

$$= \frac{k - 1}{2}. \quad (3.1.6)$$

The lowest eigenfunction is $e^{-\frac{|y|^2}{2}}$, and so from (3.1.1) we see that the lowest eigenfunction of L is $u = e^{\frac{|x|^2}{8}} e^{-\frac{|y|^2}{2}} = 1$. Therefore, the constant functions are the lowest eigenfunctions with eigenvalue $-\lambda = -1$. We see from (3.1.6) that the rest of the eigenvalues are non-negative. Note that a variation v is weighted volume preserving ($\int_{\Sigma} v dA_{\mu} = 0$) if and only if it is orthogonal to the constant functions. Therefore, we see that the hyper-planes are stable. \square

3.2 Complete Non-planar Hyper-surfaces

Let $V \subset C^{\infty}(\Sigma)$ denote the subspace of functions defined by $V \equiv \{\langle v, N \rangle | v \in \mathbb{R}^{n+1} \text{ is a constant vector}\}$. Colding-Minicozzi [10] consider functions $u \in V$ for the self-shrinkers of the mean curvature flow. They show that $L u = \frac{1}{2}u$. We also have that $u \in V$ are eigenfunctions of L for critical hyper-surfaces of the gaussian isoperimetric problem.

Lemma 3. *Let Σ be a critical hyper-surface to the gaussian isoperimetric problem, and $u \in V$. Then, we have that*

$$L u = \frac{1}{2}u. \quad (3.2.1)$$

Proof. Let $v \in \mathbb{R}^{n+1}$ be a constant vector such that $u = \langle v, N \rangle$. Consider a point fixed point $p \in \Sigma$ and geodesic normal coordinates at p . First, note that at p , we have $\nabla_i v^{T,k} = \nabla_i (g^{kl} \langle v, \partial_l \rangle) = -g^{kl} A_{il} u$. We have that $\nabla_j u = A_{jk} v^{T,k}$. Taking another derivative, we have at p that

$$\nabla_{ij}^2 u = \nabla_i A_{jk} v^{T,k} + A_{jk} \nabla_i v^{T,k}, \quad (3.2.2)$$

$$= \nabla_i A_{jk} v^{T,k} - A_{jk} A_i^k u. \quad (3.2.3)$$

Therefore, Codazzi's equation gives us that

$$\Delta u = \nabla_{v^T} H - |A|^2 u. \quad (3.2.4)$$

Since $H = \frac{1}{2}\langle x, N \rangle + C$, we have that $\nabla_i H = \frac{1}{2}\nabla_i \langle x, N \rangle = \frac{1}{2}A_{ij}x^{T,j}$. So, we have that $\nabla_{v^T} H = \frac{1}{2}A_{ij}v^{T,i}x^{T,j} = \frac{1}{2}\nabla_{x^T} \langle v, N \rangle = \frac{1}{2}\nabla_{x^T} u$. Therefore, we have

$$\Delta u = \frac{1}{2}\nabla_{x^T} u - |A|^2 u. \quad (3.2.5)$$

Hence, we have that $\mathcal{L}u = \frac{1}{2}u$. □

Next, we record some formulas that will be useful later.

Lemma 4. *For any hyper-surface Σ , $\phi \in C_0^\infty(\Sigma)$, and $f \in C^\infty(\Sigma)$, we have that*

$$\int_{\Sigma} \phi f \mathcal{L}(\phi f) dA_{\mu} = \int_{\Sigma} \phi^2 f \mathcal{L} f dA_{\mu} - \int_{\Sigma} |\nabla \phi|^2 f^2 dA_{\mu}. \quad (3.2.6)$$

If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a critical hyper-surface for the gaussian isoperimetric problem ($H = \frac{1}{2}\langle x, N \rangle + C$), then for any $\phi \in C_0^\infty(\Sigma)$ and any constant vector $v \in \mathbb{R}^{n+1}$, we have that

$$\int_{\Sigma} \phi^2 |A|^2 \langle v, N \rangle dA_{\mu} = 2 \int_{\Sigma} \phi A \langle \nabla \phi, v^T \rangle dA_{\mu}, \quad (3.2.7)$$

and

$$\mathcal{L}|x|^2 = 2n - |x|^2 - 2C \langle x, N \rangle. \quad (3.2.8)$$

Remark 2. Equation (3.2.7) measures the effect of using cut-off functions to make the following formal argument rigorous. Consider a function u with $\mathcal{L}u = \frac{1}{2}u$. So, we have $\int_{\Sigma} \mathcal{L}u dA_{\mu} = \frac{1}{2} \int_{\Sigma} u dA_{\mu}$. However, formally applying integration by parts we get $\int_{\Sigma} \mathcal{L}u dA_{\mu} = \int_{\Sigma} (\mathcal{L}u + \frac{1}{2}u + |A|^2 u) dA_{\mu} = \int_{\Sigma} [\frac{1}{2}u + |A|^2 u] dA_{\mu}$. So, formally, we have $\int_{\Sigma} |A|^2 u dA_{\mu} = 0$. The right hand side of (3.2.7) is the term we pick up from rigorously using a cut-off function.

Proof. Let us first show (3.2.6). We first work with the operator \mathcal{L} . We get

$$\int_{\Sigma} \phi f \mathcal{L}(\phi f) dA_{\mu} = \int_{\Sigma} (f^2 \phi \mathcal{L}\phi + 2\phi f \langle \nabla \phi, \nabla f \rangle + \phi^2 f \mathcal{L}f) dA_{\mu}, \quad (3.2.9)$$

$$= \int_{\Sigma} \left(f^2 \phi \mathcal{L}\phi + \frac{1}{2} \langle \nabla \phi^2, \nabla f^2 \rangle + \phi^2 f \mathcal{L}f \right) dA_{\mu}. \quad (3.2.10)$$

Then, using integration by parts we have

$$= \int_{\Sigma} \left(f^2 \phi \mathcal{L} \phi - \frac{1}{2} f^2 \mathcal{L} \phi^2 + \phi^2 f \mathcal{L} f \right) dA_{\mu}, \quad (3.2.11)$$

$$= \int_{\Sigma} \left(-f^2 |\nabla \phi|^2 + \phi^2 f \mathcal{L} f \right) dA_{\mu}. \quad (3.2.12)$$

Then we use $L = \mathcal{L} + |A|^2 + \frac{1}{2}$ to get (3.2.6).

Now, we prove (3.2.7) for the case that Σ is a critical hyper-surface for the gaussian isoperimetric problem. Therefore, we have that $L\langle v, N \rangle = \frac{1}{2}\langle v, N \rangle$. So, using integration by parts we have

$$\frac{1}{2} \int_{\Sigma} \phi^2 \langle v, N \rangle dA_{\mu} = \int_{\Sigma} \phi^2 L \langle v, N \rangle dA_{\mu}, \quad (3.2.13)$$

$$= \int_{\Sigma} \left(|A|^2 + \frac{1}{2} \right) \phi^2 \langle v, N \rangle dA_{\mu} + \int_{\Sigma} \phi^2 \mathcal{L} \langle v, N \rangle dA_{\mu}, \quad (3.2.14)$$

$$= \int_{\Sigma} \left(|A|^2 + \frac{1}{2} \right) \phi^2 \langle v, N \rangle dA_{\mu} - \int_{\Sigma} 2g^{ij} \phi \nabla_i \phi \nabla_j \langle v, N \rangle dA_{\mu}. \quad (3.2.15)$$

Therefore, we get

$$\int_{\Sigma} \phi^2 |A|^2 \langle v, N \rangle = 2 \int_{\Sigma} g^{ij} \phi \nabla_i \phi \nabla_j \langle v, N \rangle dA_{\mu}, \quad (3.2.16)$$

$$= 2 \int_{\Sigma} g^{ij} \phi \nabla_i \phi A(v^T, \partial_j) dA_{\mu}. \quad (3.2.17)$$

So we get (3.2.7).

Finally, to show (3.2.8), we note that $\Delta|x|^2 = 2n - 2\langle x, N \rangle H$. Since Σ is a critical hyper-surface of the gaussian isoperimetric problem, we have that $H = \frac{1}{2}\langle x, N \rangle + C$. Therefore, we have that $\Delta|x|^2 = 2n - \langle x, N \rangle C - |x^N|^2$. Also, note that $\nabla_{x^T}|x|^2 = 2|x^T|^2$. Hence, we get $\mathcal{L}|x|^2 = 2n - \langle x, N \rangle C - |x|^2$.

□

Now, consider the subspace $W \subset C^\infty(\Sigma)$ defined by

$$W = \{d + \langle v, N \rangle | d \in \mathbb{R} \text{ and } v \in \mathbb{R}^{n+1} \text{ is a constant vector}\}. \quad (3.2.18)$$

For any fixed $\phi \in C_0^\infty(\Sigma)$, we may consider the subspace $\phi W \subset C_0^\infty(\Sigma)$ given by the product of functions in W with the function ϕ .

Also, consider the quadratic form defined by

$$Q(u) = - \int_{\Sigma} u L u dA_{\mu}. \quad (3.2.19)$$

We show that there exists a choice of fixed $\phi \in C_0^{\infty}(\Sigma)$ such that Q is negative definite on ϕW .

Proposition 1. *Let Σ be a critical hyper-surface to the gaussian isoperimetric problem. Then, there exists $\phi \in C_0^{\infty}(\Sigma)$ such that the quadratic form Q is negative definite on ϕW where W is the sub-space of $C^{\infty}(\Sigma)$ defined by (3.2.18). Furthermore, we also have $\text{Dim } \phi W = \text{Dim } W$.*

Proof. Consider any $d + \langle v, N \rangle \in W$ with $|d|^2 + |v|^2 = 1$. For now, let us compute $Q(u\phi)$ for any $\phi \in C_0^{\infty}(\Sigma)$. Using (3.2.6) we have that

$$Q(u\phi) = - \int_{\Sigma} \phi^2 u L u dA_{\mu} + \int_{\Sigma} |\nabla \phi|^2 dA_{\mu}, \quad (3.2.20)$$

$$= - \int_{\Sigma} \phi^2 u d \left(\frac{1}{2} + |A|^2 \right) dA_{\mu} - \frac{1}{2} \int_{\Sigma} \phi^2 u \langle v, N \rangle dA_{\mu} + \int_{\Sigma} |\nabla \phi|^2 dA_{\mu}. \quad (3.2.21)$$

Then, using that $u = d + \langle v, N \rangle$ and grouping terms we obtain

$$\begin{aligned} Q(u\phi) &= - \int_{\Sigma} \phi^2 d (d + \langle v, N \rangle) \left(\frac{1}{2} + |A|^2 \right) dA_{\mu} - \frac{1}{2} \int_{\Sigma} \phi^2 (d + \langle v, N \rangle) \langle v, N \rangle dA_{\mu} \\ &\quad + \int_{\Sigma} |\nabla \phi|^2 u^2 dA_{\mu}, \end{aligned} \quad (3.2.22)$$

$$\begin{aligned} &= - \int_{\Sigma} \phi^2 \left(\frac{1}{2} d^2 + d \langle v, N \rangle + \frac{1}{2} \langle v, N \rangle^2 \right) dA_{\mu} - \int_{\Sigma} \phi^2 d^2 |A|^2 dA_{\mu} \\ &\quad - \int_{\Sigma} \phi^2 |A|^2 d \langle v, N \rangle dA_{\mu} + \int_{\Sigma} |\nabla \phi|^2 dA_{\mu}, \end{aligned} \quad (3.2.23)$$

$$\begin{aligned} &= - \frac{1}{2} \int_{\Sigma} \phi^2 u^2 dA_{\mu} - \int_{\Sigma} \phi^2 |A|^2 d^2 dA_{\mu} - \int_{\Sigma} \phi^2 |A|^2 d \langle v, N \rangle dA_{\mu} \\ &\quad + \int_{\Sigma} |\nabla \phi|^2 u^2 dA_{\mu}. \end{aligned} \quad (3.2.24)$$

Then, we use (3.2.7) and $|v|^2 \leq 1$ to get

$$\left| \int_{\Sigma} \phi^2 |A|^2 d \langle v, N \rangle dA_{\mu} \right| = 2 \left| \int_{\Sigma} \phi A (\nabla \phi, v^T) dA_{\mu} \right|, \quad (3.2.25)$$

$$\leq \int_{\Sigma} \phi^2 |A|^2 d^2 dA_{\mu} + \int_{\Sigma} |\nabla \phi|^2 dA_{\mu}. \quad (3.2.26)$$

Therefore, we have that

$$Q(u\phi) \leq -\frac{1}{2} \int_{\Sigma} \phi^2 u^2 dA_{\mu} + \int_{\Sigma} |\nabla \phi|^2 (u^2 + 1) dA_{\mu}, \quad (3.2.27)$$

$$\leq -\frac{1}{2} \int_{\Sigma} \phi^2 u^2 dA_{\mu} + 3 \int_{\Sigma} |\nabla \phi|^2 dA_{\mu}. \quad (3.2.28)$$

Next, we determine a family of cut-off functions that will be useful. We consider linear (in radius) cut-off functions defined by

$$\phi_R(r) = \begin{cases} 1 & r \leq R \\ 1 - \frac{1}{R}(r - R) & R \leq r \leq 2R \\ 0 & r \geq 2R \end{cases} \quad (3.2.29)$$

where r is the euclidean distance to the origin. Observe that $|\nabla_{\Sigma} \phi_R| \leq 1/R$.

We wish to now determine an appropriate range of R to use for our cut-off function. Note that we may not have that $\text{Dim } W = n + 1$. Let $u_i = d_i + \langle v_i, N \rangle$ be a basis for W with $d_i^2 + |v_i|^2 = 1$. Consider the set $S \subset W$ defined by $S \equiv \{e^i u_i \mid \sum_i (e^i)^2 = 1\}$. Note that since $\text{Dim } W < \infty$, we have that S is compact. Furthermore, there exists R_0 such that for all $u \in S$ we have that $u \not\equiv 0$ on $B_{R_0}(0)$. To see so, consider the case that for some sequence $R_j \rightarrow \infty$ there exists a sequence e_j^i with $\sum (e_j^i)^2 = 1$ and $e_j^i u_i \equiv 0$ on $B_{R_j}(0)$. We have that $e_j^i \rightarrow e_{\infty}^i$, and $e_{\infty}^i u_i \equiv 0$ on Σ . Then $e_{\infty}^i = 0$, and we have a contradiction.

So, it is clear that for $R \geq R_0$, we have that $\text{Dim}(\phi_R W) = \text{Dim } W$, and

$$\int_{\Sigma} \phi_R^2 u^2 dA_{\mu} \geq m_R > 0, \quad (3.2.30)$$

for all $u \in S$. Furthermore, m_R is non-decreasing in R .

Finally, since $A_{\mu}(\Sigma) < \infty$, we have that W is a finite dimensional subspace of $L_{\mu}^2(\Sigma)$. Therefore, we have that $\int_{\Sigma} |\nabla \phi_R|^2 dA_{\mu} \rightarrow 0$ uniformly for $u \in S$ as $R \rightarrow \infty$. So, from (3.2.28), we get that for some $R > R_0$ that $Q(\phi_R u) \leq -m_R/4 < 0$ for all $u \in S$. \square

Theorem 2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a non-planar critical hyper-surface for the gaussian isoperimetric problem with $\text{Index} = I$ for all compact weighted volume preserving variations. Also, we require $0 \leq I \leq n$. Then*

$$\Sigma = \Sigma_0 \times \mathbb{R}^i, \quad (3.2.31)$$

where $i \geq n + 1 - I$. In particular, there are no non-planar stable solutions.

Remark 3. The restriction we put on the index is only to highlight for which values of the index does the theorem say something meaningful.

Proof. Again consider the subspace $W \subset C^\infty(\Sigma)$ defined in (3.2.18). First, we consider the dimension of W . Let d_0 be any constant function such that $d_0 \in \text{Span}\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}$. Since $\text{Span}\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}$ is an eigenspace of L , we have that $L d_0 = \frac{1}{2} d_0$. However, since d_0 is a constant function, we also have that $L d_0 = \frac{1}{2} d_0 + |A|^2 d_0$. Hence, we have that $|A|^2 d_0 \equiv 0$. Since Σ is non-planar, we have $d_0 = 0$. Therefore, $\text{Dim } W = 1 + \text{Dim Span}\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}$.

By Proposition 1 we have for some $\phi \in C_0^\infty(\Sigma)$ that $\text{Dim}(\phi W) = \text{Dim } W$ and Q is negative definite on ϕW . Consider the functional $u \rightarrow \int_\Sigma u dA_\mu$. Considering the kernel of this functional and by counting dimensions, we see that $\text{Dim}(\phi W \cap 1^\perp) \geq \text{Dim Span}\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}$. Hence, $\text{Dim Span}\{\langle v, N \rangle\} \leq I$.

Now, consider the linear map $v \in \mathbb{R}^{n+1} \rightarrow \langle v, N \rangle \in C^\infty(\Sigma)$ and its kernel K . Let $k = \text{Dim } K$. Note that $\text{Dim Span}\{\langle v, N \rangle\} = n + 1 - k$ and so we have that $n + 1 - k \leq I$. Finally, note that Σ splits off the linear space K , and

$$\Sigma = \Sigma_0 \times K \tag{3.2.32}$$

where Σ_0 is a $(n - k)$ -dimensional slice of Σ . □

3.3 Stable Critical Hyper-surfaces in a Ball

Now, we consider two-sided hyper-surfaces Σ with boundary $\partial\Sigma$; specifically we consider $\Sigma \subset B_R(0)$ with $\partial\Sigma \subset \partial B_R(0)$. That is, Σ is contained in a ball centered at the origin, and $\partial\Sigma$ is contained in the boundary of the ball. We require that Σ be proper immersed, but we do not directly require any volume growth conditions.

Like in the complete case, the functions $\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}$ will play an important role. Often when working with stable minimal hyper-surfaces, it is productive to use certain test functions with a stability inequality. For the minimal case, one is free to construct any compactly supported test function that one wishes to use. For the gaussian isoperimetric problem, we are limited by the fact that the stability inequality $0 \geq -\int_\Sigma u L u dA_\mu$ only applies to those compact variations u that preserve weighted volume. The functions $\{\langle v, N \rangle\}_{v \in \mathbb{R}^{n+1}}$ will serve the purpose of aiding us in recovering a form of the stability inequality that applies to more convenient compact variations. The price we pay for a more general stability inequality is that our new inequality is more cumbersome

to work with.

We will need to make use of an average of the normal N that depends on a choice of cut-off function ϕ with $\text{suppt } \phi \subset \subset B_R(0)$. Let $\phi \in C_0^\infty(\Sigma)$ and $\int_\Sigma \phi dA_\mu \neq 0$. We let

$$N_\phi = \frac{\int_\Sigma \phi N dA_\mu}{\int_\Sigma \phi dA_\mu}. \quad (3.3.1)$$

We now show a modified form of the stability inequality that applies to more convenient variations.

Lemma 5. *Let $\Sigma \subset \mathbb{R}^{n+1}$ (possibly with boundary) be a stable critical hyper-surface of the gaussian isoperimetric problem. Let $\phi \in C_0^\infty(\Sigma)$ ($\text{suppt}(\phi) \cap \partial\Sigma = \emptyset$) such that $\int_\Sigma \phi dA_\mu \neq 0$. Then, we have that*

$$\int_\Sigma \phi^2 |N - N_\phi|^2 dA_\mu + |N_\phi|^2 \int_\Sigma \phi^2 |A|^2 dA_\mu \leq 4(n+2) \int_\Sigma |\nabla \phi|^2 dA_\mu. \quad (3.3.2)$$

Proof. Note that for any $v \in \mathbb{R}^{n+1}$ we have that $\int_\Sigma \phi \langle v, N - N_\phi \rangle dA_\mu = 0$. Therefore, we have that $Q(\phi \langle v, N - N_\phi \rangle) \geq 0$ and we use (3.2.6) to get that

$$\int_\Sigma \phi^2 \langle v, N - N_\phi \rangle L \langle v, N - N_\phi \rangle dA_\mu \leq \int_\Sigma |\nabla \phi|^2 \langle v, N - N_\phi \rangle^2 dA_\mu. \quad (3.3.3)$$

Now, we use that $L \langle v, N \rangle = \frac{1}{2} \langle v, N \rangle$ and $L \langle v, N_\phi \rangle = (\frac{1}{2} + |A|^2) \langle v, N_\phi \rangle$ to get that

$$\begin{aligned} & \int_\Sigma \phi^2 \langle v, N - N_\phi \rangle L \langle v, N - N_\phi \rangle dA_\mu \\ &= \frac{1}{2} \int_\Sigma \phi^2 \langle v, N - N_\phi \rangle^2 dA_\mu - \int_\Sigma \phi^2 |A|^2 \langle v, N_\phi \rangle \langle v, N - N_\phi \rangle dA_\mu. \end{aligned} \quad (3.3.4)$$

We apply a Cauchy inequality to (3.2.7) to get

$$\langle v, N_\phi \rangle \int_\Sigma |A|^2 \phi^2 \langle v, N \rangle \leq \frac{1}{2} \int_\Sigma \phi^2 \langle v, N_\phi \rangle^2 |A|^2 dA_\mu + 2 \int_\Sigma |\nabla \phi|^2 |v^T|^2 dA_\mu. \quad (3.3.5)$$

Using (3.3.3), (3.3.4), and (3.3.5) we get

$$\int_\Sigma \phi^2 \langle v, N - N_\phi \rangle^2 dA_\mu + \langle v, N_\phi \rangle^2 \int_\Sigma \phi^2 |A|^2 dA_\mu$$

$$\leq 2 \int_{\Sigma} |\nabla \phi|^2 \langle v, N - N_{\phi} \rangle^2 dA_{\mu} + 4 \int_{\Sigma} |\nabla \phi|^2 |v^T|^2 dA_{\mu}. \quad (3.3.6)$$

Now, summing over v for an orthonormal frame of \mathbb{R}^{n+1} , we get

$$\begin{aligned} & \int_{\Sigma} \phi^2 |N - N_{\phi}|^2 dA_{\mu} + |N_{\phi}|^2 \int_{\Sigma} \phi^2 |A|^2 dA_{\mu} \\ & \leq 2 \int_{\Sigma} |\nabla \phi|^2 |N - N_{\phi}|^2 dA_{\mu} + 4n \int_{\Sigma} |\nabla \phi|^2 dA_{\mu}. \end{aligned} \quad (3.3.7)$$

Using that $|N - N_{\phi}|^2 \leq 4$, we then get the lemma. □

3.3.1 Integral Curvature Estimates

Now, we will use our modified version of the stability inequality (3.3.2) to derive integral decay-estimates for $|A|^2$. These estimates depend on being in a large enough ball and on reasonable euclidean area growth conditions for the hyper-surface.

During the course of proving our estimates, we will need to make use of arbitrary constants. We will often need to take constants of a larger size to match the statement of another lemma or proposition. Therefore, everytime we make use of arbitrary constants, we write our statements in such a way as to guarantee their validity if we increase the size of those constants.

Proposition 2. *Let $\Sigma \subset B_R(0) \subset \mathbb{R}^{n+1}$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{1}{2} \langle x, N \rangle + C$. Also let $0 < \theta < 1$. Then, there exists a constant D_n such that if $\theta R > D_n \mathcal{O}_C$ and the euclidean area $A_E(\Sigma \cap B_R(0)) < D_n^{-1} e^{-D_n \mathcal{O}_C^2} \mathcal{O}_C^n (1 - \theta)^2 R^2 e^{\theta^2 R^2/4}$ then*

$$\int_{B_{\theta R}(0)} |A|^2 dA_{\mu} \leq D_n e^{D_n \mathcal{O}_C^2} \mathcal{O}_C^{-n} \frac{A_E(\Sigma)^2}{(1 - \theta)^2} R^{-2} e^{-\theta R^2/4}. \quad (3.3.8)$$

Proof. For brevity and clarity of notation, throughout this proof we will use $A_E(\Sigma) \equiv A_E(\Sigma \cap B_R(0))$ to be the euclidean area of Σ inside $B_R(0)$.

Again, we make use of a linear cut-off function:

$$\phi(r) = \begin{cases} 1 & r \leq \theta R \\ 1 - \frac{r - \theta R}{(1 - \theta)R} & \theta R \leq r \leq R \\ 0 & r \geq R. \end{cases} \quad (3.3.9)$$

Note that $|\nabla\phi| \leq \frac{1}{(1-\theta)R}$.

Using the modified stability inequality (3.3.2), we see that

$$\begin{aligned} \int_{\Sigma} \phi^2 dA_{\mu} - 2 \left(\int_{\Sigma} \phi^2 dA_{\mu} \right) |N_{\phi}| + \left(\int_{\Sigma} \phi^2 (|A|^2 + 1) dA_{\mu} \right) |N_{\phi}|^2 \\ \leq B_n \frac{A_E(\Sigma)}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4}. \end{aligned} \quad (3.3.10)$$

We see that the left hand side of (3.3.10) is quadratic in $|N_{\phi}|$. Therefore, since any quadratic $au^2 + bu + c$ with $a > 0$ satisfies $au^2 + bu + c \geq c - \frac{b^2}{4a}$, we get that

$$\int_{\Sigma} \phi^2 dA_{\mu} - \frac{\left(\int_{\Sigma} \phi^2 dA_{\mu} \right)^2}{\int_{\Sigma} \phi^2 (|A|^2 + 1) dA_{\mu}} \leq B_n \frac{A_E(\Sigma)}{(1-\theta)R^2} e^{-\theta^2 R^2/4}. \quad (3.3.11)$$

Combining terms on the left hand side of (3.3.11), we get

$$\frac{\left(\int_{\Sigma} \phi^2 dA_{\mu} \right) \left(\int_{\Sigma} \phi^2 |A|^2 dA_{\mu} \right)}{\int_{\Sigma} \phi^2 (1 + |A|^2) dA_{\mu}} \leq B_n \frac{A_E(\Sigma)}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4}. \quad (3.3.12)$$

Cross multiplying in (3.3.12) and grouping like terms we obtain

$$\begin{aligned} \left(\int_{\Sigma} \phi^2 dA_{\mu} - B_n \frac{A_E(\Sigma)}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4} \right) \int_{\Sigma} \phi^2 |A|^2 dA_{\mu} \\ \leq B_n \frac{A_E(\Sigma)}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4} \int_{\Sigma} \phi^2 dA_{\mu}, \end{aligned} \quad (3.3.13)$$

$$\leq B_n \frac{A_E(\Sigma)^2}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4}. \quad (3.3.14)$$

To turn (3.3.14) into an estimate for $\int_{\Sigma} \phi^2 |A|^2 dA_{\mu}$, we need a lower bound for $\int_{\Sigma} \phi^2 dA_{\mu}$. To create such a bound, it is sufficient to put a lower bound on the area around the point on Σ where the minimum for $|x|$ is achieved. By (A.0.82), it is sufficient to provide control over $|H|$ in a region around the point where the minimum of $|x|$ is achieved.

Let $p \in \Sigma$ be a point where the minimum of $|x|$ is achieved. Note that at such a point $\mathcal{L}|x|^2 \geq 0$.

Therefore, from (3.2.8), we have at p that

$$0 \leq 2n - |p|^2 - 2C\langle p, N \rangle, \quad (3.3.15)$$

$$\leq 2n - |p|^2 + 2|C||p|. \quad (3.3.16)$$

Therefore, we have that

$$|p| \leq |C| + \sqrt{C^2 + 2n} \leq D_{n,1}\mathcal{O}_C. \quad (3.3.17)$$

So, for any radius R_0 such that $D_{n,1}\mathcal{O}_C \leq R_0 \leq 2D_{n,1}\mathcal{O}_C$, we have that $\Sigma \cap B_{R_0}(0) \neq \emptyset$. Since $H = \frac{1}{2}\langle x, N \rangle + C$, we have that $|H| \leq D_{n,2}\mathcal{O}_C$ on $B_{R_0}(0)$. So, by (A.0.82), we have that there is a constant $D_{n,3}$ such that if $D_{n,3}\mathcal{O}_C \leq R_0 \leq 2D_{n,3}\mathcal{O}_C$, then we have that $A(\Sigma \cap B_{R_0}(0)) \geq D_{n,3}^{-1}e^{-D_{n,3}\mathcal{O}_C^2}\mathcal{O}_C^n$. Therefore, there is a constant $D_{n,4}$ such that if $\theta R > D_{n,4}\mathcal{O}_C$, then $\int_\Sigma \phi^2 dA_\mu \geq D_{n,4}^{-1}e^{-D_{n,4}\mathcal{O}_C^2}\mathcal{O}_C^n$.

So, if we require $\theta R > D_{n,4}\mathcal{O}_C$, then (3.3.14) becomes

$$\begin{aligned} & \left(D_{n,4}^{-1}e^{-D_{n,4}\mathcal{O}_C^2}\mathcal{O}_C^n - B_n \frac{A_E(\Sigma)}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4} \right) \int_\Sigma \phi^2 |A|^2 dA_\mu, \\ & \leq B_n \frac{A_E(\Sigma)^2}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4}. \end{aligned} \quad (3.3.18)$$

Now, if we also require that $A_E(\Sigma) \leq \frac{1}{2}D_{n,4}^{-1}e^{-D_{n,4}\mathcal{O}_C^2}\mathcal{O}_C^n(1-\theta)^2 R^2 e^{\theta^2 R^2/4}$, then we obtain that

$$D_{n,4}^{-1}e^{-D_{n,4}\mathcal{O}_C^2}\mathcal{O}_C^n \int_\Sigma \phi^2 |A|^2 dA_\mu \leq D_{n,4} \frac{A_E(\Sigma)^2}{(1-\theta)^2 R^2} e^{-\theta^2 R^2/4}. \quad (3.3.19)$$

From (3.3.19) we have the lemma. □

3.3.2 Pointwise Curvature Estimates for Three-dimensional Case

First we need to make use of a Simons' type inequality that is true for critical hyper-surfaces in any dimension.

Lemma 6. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be any critical hyper-surface of the gaussian isoperimetric problem with $H = \frac{1}{2}\langle x, N \rangle + C$. Then,*

$$LA = A + CA^2, \quad (3.3.20)$$

and we have the estimate (at least weakly)

$$\mathcal{L}|A| \geq - \left(1 + \frac{C^2}{2} \right) |A|^3. \quad (3.3.21)$$

Proof. We first show (3.3.20). Consider a point $p \in \Sigma$ and geodesic normal coordinates centered at p . From the symmetry of ∇A and the Leibniz rule for the curvature tensor R , we have that

$$\Delta A_{jk} = \nabla_{ii}^2 A_{jk} = \nabla_{ij}^2 A_{ik}, \quad (3.3.22)$$

$$= \nabla_{ji}^2 A_{ik} + R_{ij} A_{ik}, \quad (3.3.23)$$

$$= \nabla_{ji}^2 A_{ik} - R_{iji}{}^l A_{lk} - R_{ijk}{}^l A_{il}. \quad (3.3.24)$$

Next, using Gauss' equation and the symmetries of ∇A , we have that

$$\Delta A_{jk} = \nabla_{ji}^2 A_{ik} + (A_{ii} A_j{}^l - A_{ij} A_i{}^l) A_{lk} + (A_{ik} A_j{}^l - A_{jk} A_i{}^l) A_{il}, \quad (3.3.25)$$

$$= \nabla_{jk}^2 H + H A_{jk}^2 - |A|^2 A_{jk}. \quad (3.3.26)$$

Now, we use that $H = \frac{\langle x, N \rangle}{2} + C$ to get that

$$\nabla_k H = \frac{1}{2} A(\partial_k, x^T). \quad (3.3.27)$$

So, using a Leibniz formula and the symmetries of ∇A , we see that

$$\nabla_{jk}^2 H = \frac{1}{2} \nabla_j A(\partial_k, x^T) + \frac{1}{2} A_{jk} - \frac{1}{2} \langle x, N \rangle A_{jk}^2, \quad (3.3.28)$$

$$= \frac{1}{2} \nabla_{x^T} A(\partial_j, \partial_k) + \frac{1}{2} A_{jk} - \frac{1}{2} \langle x, N \rangle A_{jk}^2. \quad (3.3.29)$$

Using (3.3.26) and (3.3.29) we have that

$$\mathcal{L} A_{jk} = A_{jk} + H A_{jk}^2 - \frac{\langle x, N \rangle}{2} A_{jk}^2, \quad (3.3.30)$$

$$= A_{jk} + C A_{jk}^2. \quad (3.3.31)$$

Therefore, we have (3.3.20)

Next, we show (3.3.21). Note that by (3.3.20), we have that

$$\mathcal{L} A = \frac{1}{2} A + C A^2 - |A|^2 A. \quad (3.3.32)$$

Now, note for any function f , by a straightforward calculation we have that

$$\mathcal{L} \sqrt{f} = \frac{\mathcal{L} f}{2\sqrt{f}} - \frac{|\nabla f|^2}{4f^{3/2}}. \quad (3.3.33)$$

So, we see that

$$\mathcal{L}|A| = \frac{\mathcal{L}|A|^2}{2|A|} - \frac{|\nabla|A|^2|^2}{4|A|^3}, \quad (3.3.34)$$

$$= \frac{\mathcal{L}|A|^2}{2|A|} - \frac{|\nabla|A|^2|^2}{|A|}, \quad (3.3.35)$$

$$= \frac{\langle A, \mathcal{L}A \rangle + |\nabla A|^2 - |\nabla|A|^2|^2}{|A|}. \quad (3.3.36)$$

From (3.3.32), we get that

$$\mathcal{L}|A| = \frac{\frac{1}{2}|A|^2 + C \operatorname{Tr} A^3 - |A|^4 + |\nabla A|^2 - |\nabla|A|^2|^2}{|A|}, \quad (3.3.37)$$

$$= \frac{1}{2}|A| + \frac{C}{|A|} \operatorname{Tr} A^3 - |A|^3 + \frac{|\nabla A|^2 - |\nabla|A|^2|^2}{|A|}, \quad (3.3.38)$$

$$\geq \frac{1}{2}|A| - |C||A|^2 - |A|^3. \quad (3.3.39)$$

Then, applying a Cauchy inequality we get

$$\mathcal{L}|A| \geq -\left(1 + \frac{C^2}{2}\right)|A|^3, \quad (3.3.40)$$

which is (3.3.21). \square

Now, instead of using a Choi-Schoen [8] type argument as in McGonagle-Ross [27], we will obtain point-wise decay estimates for $|A|$ based on a de Giorgi-Moser-Nash type iteration argument [3, 4, 19]. The main technical point of our argument is that we wish to create estimates that give us exponential decay on any sub-ball $B_{\theta R}(0)$ for $0 < \theta < 1$. To do so, we must be careful with where we put our exponential terms in each step of the proof of the estimates.

Next, we put the Michael-Simon Sobolev inequality for hyper-surfaces [28] into a form suitable for working with the weighted area measure dA_μ for $\Sigma^2 \subset \mathbb{R}^3$.

Lemma 7. *Let $\Sigma^2 \subset \mathbb{R}^3$ be any critical hyper-surface of the gaussian isoperimetric problem, and let $f \in C_0^1(\Sigma \cap B_r(0))$. Then, we have a Sobolev-type inequality:*

$$\int_{B_r(0)} f^2 dA_\mu \leq D e^{\frac{r^2}{4}} \left[\left(\int_{B_r(0)} |\nabla f| dA_\mu \right)^2 + (1 + r^2 + C^2) \left(\int_{B_r(0)} |f| dA_\mu \right)^2 \right]. \quad (3.3.41)$$

Here, D is a constant independent of both f and Σ .

Proof. The Michael-Simon Sobolev inequality [28] for hyper-surfaces $\Sigma^2 \subset \mathbb{R}^3$ states that for any

$g \in C_0^1(\Sigma)$, we have that

$$\left(\int_{B_r(0)} g^2 dA \right)^{\frac{1}{2}} \leq D_1 \left(\int_{B_r(0)} |\nabla g| dA + \int_{B_r(0)} |gH| dA \right). \quad (3.3.42)$$

Now, for any $f \in C_0^1(\Sigma \cap B_r(0))$, consider the function $g = fe^{-|x|^2/8}$. Putting g into (3.3.42) we get

$$\left(\int_{B_r(0)} f^2 dA_\mu \right)^{\frac{1}{2}} \leq D_1 \left[\int_{B_r(0)} |\nabla f| e^{|x|^2/8} dA_\mu + \int_{B_r(0)} |f| \left(\frac{|x|}{4} + |H| \right) e^{|x|^2/8} dA_\mu \right], \quad (3.3.43)$$

$$\leq D_2 e^{\frac{r^2}{8}} \left[\int_{B_r(0)} |\nabla f| dA_\mu + (1 + r + |C|) \int_{B_r(0)} |f| dA_\mu \right]. \quad (3.3.44)$$

Squaring both sides and using another estimate gives us inequality (3.3.41). \square

Now we may use the Simons inequality (3.3.21) and integration by parts to give integral estimates for $|\nabla \phi|A|^k|^2$. For clarity of notation, we will use $u \equiv |A|$.

For clarity of keeping track of orders of terms, we will also make use of the following notation:

$$\mathcal{O}_r \equiv 1 + r, \quad (3.3.45)$$

$$\mathcal{O}_C \equiv 1 + |C|, \quad (3.3.46)$$

$$\mathcal{O}_\phi \equiv 1 + \|\phi\|_{C^1}. \quad (3.3.47)$$

Here, $\|\phi\|_{C^1} \equiv \sup |\phi| + \sup |\nabla \phi|$.

Lemma 8. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{\langle x, N \rangle}{2} + C$. Also, let $\phi \in C_0^1(\Sigma)$, $k \geq 1$, and $u \equiv |A|$. Then we have that*

$$\int_{\Sigma} |\nabla(\phi u^k)|^2 dA_\mu \leq \int_{\Sigma} |\nabla \phi|^2 u^{2k} dA_\mu + k \left(1 + \frac{C^2}{2} \right) \int_{\Sigma} \phi^2 u^{2k+2} dA_\mu. \quad (3.3.48)$$

Proof. First, we note that

$$\begin{aligned} \int_{\Sigma} |\nabla(\phi u^k)|^2 dA_\mu &= \int_{\Sigma} |\nabla \phi|^2 u^{2k} dA_\mu + k^2 \int_{\Sigma} \phi^2 u^{2k-2} |\nabla u|^2 dA_\mu \\ &\quad + 2k \int_{\Sigma} \phi u^{2k-1} \langle \nabla \phi, \nabla u \rangle dA_\mu. \end{aligned} \quad (3.3.49)$$

Next, we note that

$$\langle \nabla(\phi^2 u^{2k-1}), \nabla u \rangle = 2\phi u^{2k-1} \langle \nabla \phi, \nabla u \rangle + (2k-1)\phi^2 u^{2k-2} |\nabla u|^2. \quad (3.3.50)$$

Using (3.3.49) and (3.3.50), we get that

$$\begin{aligned} \int_{\Sigma} |\nabla(\phi u^k)|^2 dA_{\mu} &= \int_{\Sigma} |\nabla \phi|^2 u^{2k} dA_{\mu} + (k-k^2) \int_{\Sigma} \phi^2 u^{2k-2} |\nabla u|^2 dA_{\mu} \\ &\quad + k \int_{\Sigma} \langle \nabla(\phi^2 u^{2k-1}), \nabla u \rangle dA_{\mu}. \end{aligned} \quad (3.3.51)$$

Now, for integers $k \geq 0$, we have $k - k^2 \leq 0$. So using this, integration by parts, and the Simons-type inequality (3.3.21), we have that (3.3.51) becomes

$$\int_{\Sigma} |\nabla(\phi u^k)|^2 dA_{\mu} \leq \int_{\Sigma} |\nabla \phi|^2 u^{2k} dA_{\mu} + k \left(1 + \frac{C^2}{2}\right) \int_{\Sigma} \phi^2 u^{2k+2} dA_{\mu}. \quad (3.3.52)$$

□

Now, we may use the Sobolev inequality (3.3.41) and the inequality (3.3.48) from Lemma 8 to get better estimates. Here, we will find it convenient to use $\|f\|_p$ to be the weighted L^p norm

$$\|f\|_p = \left(\int_{\Sigma} f^p dA_{\mu} \right)^{\frac{1}{p}}, \quad (3.3.53)$$

and on a ball $B_r(0)$,

$$\|f\|_{p,r} = \left(\int_{\Sigma \cap B_r(0)} f^p dA_{\mu} \right)^{\frac{1}{p}}. \quad (3.3.54)$$

Lemma 9. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{\langle x, N \rangle}{2} + C$. Also, let $u \equiv |A|$ and $\phi \in C_0^1(\Sigma \cap B_r(0))$. We have that*

$$\|\phi u^k\|_4^4 \leq D e^{\frac{r^2}{4}} \|\phi u^k\|_2^2 \left(\|u^k \nabla \phi\|_2^2 + \mathcal{O}_r^2 \mathcal{O}_C^2 \|\phi u^k\|_2^2 + \mathcal{O}_C^2 k \|\phi u^{k+1}\|_2^2 \right). \quad (3.3.55)$$

Here, D is independent of Σ and ϕ .

Proof. We use $f = \phi^2 u^{2k}$ in the Sobolev inequality (3.3.41) to get that

$$\|\phi u^k\|_4^4 \leq D_1 e^{\frac{r^2}{4}} \left[\left(\int_{B_r(0)} |\phi| u^k |\nabla(\phi u^k)| dA_\mu \right)^2 + (1 + r^2 + C^2) \left(\int_{B_r(0)} \phi^2 u^{2k} \right)^2 \right]. \quad (3.3.56)$$

Using Holder's Inequality, we have that

$$\|\phi u^k\|_4^4 \leq D_1 e^{\frac{r^2}{4}} \|\phi u^k\|_2^2 (\|\nabla(\phi u^k)\|_2^2 + \mathcal{O}_r^2 \mathcal{O}_C^2 \|\phi u^k\|_2^2). \quad (3.3.57)$$

Then, using (3.3.48), we get that

$$\|\phi u^k\|_4^4 \leq D_2 e^{\frac{r^2}{4}} \|\phi u^k\|_2^2 (\|u^k \nabla \phi\|_2^2 + \mathcal{O}_C^2 k \|\phi u^{k+1}\|_2^2 + \mathcal{O}_r^2 \mathcal{O}_C^2 \|\phi u^k\|_2^2). \quad (3.3.58)$$

□

Now, we establish a recursion relation for some of the $L_\mu^p(\Sigma)$ norms on decreasing balls.

Lemma 10. *Let $\Sigma^2 \subset B_R(0) \subset \mathbb{R}^3$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{\langle x, N \rangle}{2} + C$ and $\partial \Sigma \subset \partial B_R(0)$. Also, let $u \equiv |A|$.*

There exists a constant D such that if $B_r(0) \subset B_{r+h}(0) \subset B_R(0)$, then we have that

$$\|u\|_{4k,r} \leq D^{\frac{1}{4k}} \mathcal{O}_r^{\frac{1}{2k}} \mathcal{O}_C^{\frac{3}{4k}} e^{\frac{(r+h)^2}{16k}} \left(1 + \frac{1}{h}\right)^{\frac{1}{2k}} \left(1 + k^{\frac{3}{2}} e^{\frac{(r+h)^2}{8}} \|u\|_{6,r+h}^3\right)^{\frac{1}{4k}} \|u\|_{2k,r+h}. \quad (3.3.59)$$

Remark 4. *Note that in (3.3.59), we have left an exponential order $e^{\frac{(r+h)^2}{8}}$ with $\|u\|_{6,r+h}^3$. The importance of this is that later we will show that $\|u\|_{6,r+h}^3$ may be estimated using $\|u\|_{2,R}$, and this estimate will cancel some of the exponential order in $e^{\frac{(r+h)^2}{8}} \|u\|_{6,r+h}^3$. This order cancellation is necessary to obtain a final point-wise estimate with exponential decay for $B_{\theta R}(0)$ and any $0 < \theta < 1$.*

Proof. First, consider a cut-off function ϕ with $\phi \equiv 1$ on $B_r(0)$, $\phi \equiv 0$ on $B_R(0) \setminus B_{r+h}(0)$, and $|\nabla \phi| \leq \frac{2}{h}$.

Using Holder's inequality, we may estimate

$$\|\phi u^{k+1}\|_2^2 \leq \|\phi u^k\|_3^2 \left(\int_{B_{r+h}(0)} u^6 dA_\mu \right)^{\frac{1}{3}}. \quad (3.3.60)$$

Using log-concavity interpolation for L^p spaces, we get that

$$\|\phi u^{k+1}\|_2^2 \leq \|u\|_{6,r+h}^2 \|\phi u^k\|_2^{\frac{2}{3}} \|\phi u^k\|_4^{\frac{4}{3}}. \quad (3.3.61)$$

So using equation (3.3.61) and a Cauchy ϵ -inequality of the form $ab \leq \frac{2\epsilon^{\frac{3}{2}}a^{\frac{3}{2}}}{3} + \frac{b^3}{3\epsilon^3}$, we have that

$$D_1 e^{\frac{(r+h)^2}{4}} \|\phi u^k\|_2^2 \mathcal{O}_C^2 k \|\phi u^{k+1}\|_2^2 \leq D_2 e^{\frac{3(r+h)^2}{8}} \mathcal{O}_C^3 k^{\frac{3}{2}} \|u\|_{6,r+h}^3 \|\phi u^k\|_2^4 + \frac{1}{2} \|\phi u^k\|_4^4. \quad (3.3.62)$$

So, then (3.3.55) gives us that

$$\|\phi u^k\|_4^4 \leq D_3 \mathcal{O}_r^2 \mathcal{O}_C^3 \mathcal{O}_\phi^2 e^{\frac{(r+h)^2}{4}} \left(1 + k^{\frac{3}{2}} e^{\frac{(r+h)^2}{8}} \|u\|_{6,r+h}^3 \right) \|\phi u^k\|_2^4. \quad (3.3.63)$$

Then raising to the $\frac{1}{4k}$ power we get (3.3.59). \square

We may now use iteration to prove an L^∞ bound.

Proposition 3. *Let $\Sigma^2 \subset B_R(0) \subset \mathbb{R}^3$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{\langle x, N \rangle}{2} + C$ and $\partial \Sigma \subset \partial B_R(0)$. Also, let $u \equiv |A|$.*

There exists a constant D such that for any $B_r(0) \subset B_R(0)$, we have that

$$\sup_{B_r(0)} |A| \leq D \mathcal{O}_R \mathcal{O}_C^{\frac{3}{2}} \left(1 + \frac{1}{R-r} \right) e^{\frac{1}{8}(\frac{1}{3}r^2 + \frac{2}{3}R^2)} \left(1 + e^{\frac{R^2}{16}} \|u\|_{6,R}^{\frac{3}{2}} \right) \|u\|_{2,R}. \quad (3.3.64)$$

Proof. We define a sequence r_i , k_i , and h_i for $i = 0, 1, 2, \dots$ in order to create a decreasing sequence of balls between $B_r(0)$ and $B_R(0)$. We let $r_0 = R$ and $r_{i+1} = r_i - 2^{-i-1}(R - r)$. Note that $r < r_i \leq R$ and $r_i \searrow r$. Letting h_i be the appropriate difference in radii for (3.3.59), we must have $h_i = 2^{-i-1}(r - R)$. Finally, take $k_i = 2^i$. See Figure 3.1 below.

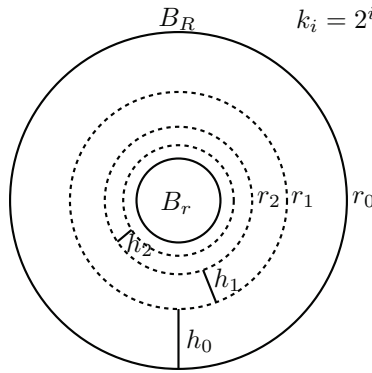


Figure 3.1: Set up of the iteration

We then have that (3.3.59) gives us that

$$\begin{aligned} \|u\|_{2^{i+2},r} &\leq \|u\|_{2,R} \left[D_1 \mathcal{O}_R^{\frac{1}{2}} \mathcal{O}_C^{\frac{3}{4}} \left(1 + \frac{1}{\sqrt{R-r}} \right) \left(1 + e^{\frac{R^2}{32}} \|u\|_{6,R}^{\frac{3}{4}} \right) \right]^{\sum_{j=0}^{\infty} \frac{1}{2^j}} \\ &\quad \times \prod_{i=0}^{\infty} e^{\frac{1}{16} \left(\frac{(r+2^{-i}(R-r))^2}{2^i} \right)} (1 + 2^{i+1})^{\frac{1}{2^i}}. \end{aligned} \quad (3.3.65)$$

We have that

$$\prod_{i=0}^{\infty} e^{\frac{1}{16} \left(\frac{(r+2^{-i}(R-r))^2}{2^i} \right)} = e^{\frac{1}{16} \left(\frac{10}{21} r^2 + \frac{8}{21} rR + \frac{8}{7} R^2 \right)}, \quad (3.3.66)$$

$$\leq e^{\frac{1}{8} \left(\frac{1}{3} r^2 + \frac{2}{3} R^2 \right)}. \quad (3.3.67)$$

Also, note that

$$\log \left(\prod_{j=0}^{\infty} (1 + 2^{j+1})^{\frac{1}{2^j}} \right) = \sum_{j=0}^{\infty} \frac{\log(1 + 2^{j+1})}{2^j}, \quad (3.3.68)$$

which converges by a simple ratio test. For this note that $\lim_{i \rightarrow \infty} \frac{\log(1+2^{i+2})}{\log(1+2^{i+1})} = \lim_{x \rightarrow \infty} \frac{\log(1+2x)}{\log(1+x)} = 1$.

Therefore, we get that

$$\|u\|_{2^{i+2},r} \leq D_2 \mathcal{O}_R \mathcal{O}_C^{\frac{3}{2}} \left(1 + \frac{1}{R-r} \right) e^{\frac{1}{8} \left(\frac{1}{3} r^2 + \frac{2}{3} R^2 \right)} \left(1 + e^{\frac{R^2}{16}} \|u\|_{6,R}^{\frac{3}{2}} \right) \|u\|_{2,R}. \quad (3.3.69)$$

Taking $i \rightarrow \infty$ we obtain an L^∞ norm. □

To get complete estimates, we will need to estimate the L_μ^6 and L_μ^4 norms of u . Note that plugging $k = 1$ and $k = \frac{3}{2}$ into (3.3.55) won't give us direct estimates for L_μ^4 and L_μ^6 in terms of lower powers. To get around this, we make an assumption on the smallness of the L_μ^2 norm of u . First, we find an estimate for L_μ^4 .

Lemma 11. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{\langle x, N \rangle}{2} + C$. Also let $R \geq 0$ and $\phi \in C_0^1(\Sigma \cap B_R(0))$. Then, there exists a constant D (independent of Σ , C , and R) such that if*

$$D \mathcal{O}_C^2 e^{\frac{R^2}{4}} \int_{B_R(0)} u^2 dA_\mu \leq 1, \quad (3.3.70)$$

then we have the estimates

$$\int_{B_R(0)} \phi^4 u^4 dA_\mu \leq \mathcal{O}_\phi^2 \mathcal{O}_R^2 \int_{B_R(0)} \phi^2 u^2 dA_\mu, \quad (3.3.71)$$

$$\int_{B_R(0)} |\nabla(\phi^2 u)|^2 dA_\mu \leq 5 \mathcal{O}_\phi^2 \mathcal{O}_R^2 \mathcal{O}_C^2 \int_{B_R(0)} u^2 dA_\mu. \quad (3.3.72)$$

Remark 5. One of the purposes of the sufficient condition (3.3.70) is to cancel orders of exponential terms. However, note that the order of $e^{\frac{R^2}{4}}$ is of opposite order of $e^{-\frac{\theta^2 R^2}{4}}$ in (3.3.8) as $\theta \rightarrow 1$. It is for this reason, that we can not increase the order of the exponential term in (3.3.70) if we want reasonable sufficient conditions in our theorems for any $0 < \theta < 1$.

Proof. Put $f = \phi^2 u^2$ into the Sobolev inequality (3.3.41) to get

$$\|\phi u\|_4^4 \leq D_1 e^{\frac{R^2}{4}} \left[\left(\int_{B_R(0)} \phi u |\nabla(\phi u)| dA_\mu \right)^2 + \mathcal{O}_R^2 \mathcal{O}_C^2 \left(\int_{B_R(0)} \phi^2 u^2 dA_\mu \right)^2 \right]. \quad (3.3.73)$$

When we use (3.3.48) to estimate our derivative terms, we will inevitably need to deal with u^4 appearing on the right hand side of our estimate. In order to absorb the u^4 term into the left hand side, we will need to make sure the power of ϕ matches. So, we use that $\phi u |\nabla(\phi u)| \leq u |\nabla(\phi^2 u)| + \phi |\nabla \phi| u^2$. Therefore, by Holder's inequality we have

$$\|\phi u\|_4^4 \leq D_2 e^{\frac{R^2}{4}} \left(\int_{B_R(0)} u^2 dA_\mu \right) \left[\int_{B_R(0)} |\nabla(\phi^2 u)|^2 dA_\mu + \mathcal{O}_\phi^2 \mathcal{O}_R^2 \mathcal{O}_C^2 \int_{B_R(0)} \phi^2 u^2 dA_\mu \right]. \quad (3.3.74)$$

Now, from (3.3.48), we have that

$$\int_{B_R(0)} |\nabla(\phi^2 u)|^2 dA_\mu \leq 4 \|\phi\|_{C^1}^2 \int_{B_R(0)} \phi^2 u^2 dA_\mu + \left(1 + \frac{C^2}{2} \right) \int_{B_R(0)} \phi^4 u^4 dA_\mu. \quad (3.3.75)$$

Therefore, we get that

$$\|\phi u\|_4^4 \leq D_3 \mathcal{O}_C^2 e^{\frac{R^2}{4}} \left(\int_{B_R(0)} u^2 dA_\mu \right) \left[\int_{B_R(0)} \phi^4 u^4 dA_\mu + \mathcal{O}_\phi^2 \mathcal{O}_R^2 \int_{B_R(0)} \phi^2 u^2 dA_\mu \right]. \quad (3.3.76)$$

Therefore, we see that there is a constant D_4 such that if

$$\mathcal{O}_C^2 e^{\frac{R^2}{4}} \int_{B_R(0)} u^2 dA_\mu \leq D_4^{-1}, \quad (3.3.77)$$

then we have that

$$\|\phi u\|_4^4 \leq \mathcal{O}_\phi^2 \mathcal{O}_R^2 \int_{B_R(0)} \phi^2 u^2 dA_\mu. \quad (3.3.78)$$

This gives us (3.3.71).

Then, using (3.3.75), we get that

$$\int_{B_R(0)} |\nabla(\phi^2 u)|^2 dA_\mu \leq 5 \mathcal{O}_\phi^2 \mathcal{O}_R^2 \mathcal{O}_C^2 \int_{B_R(0)} \phi^2 u^2 dA_\mu. \quad (3.3.79)$$

This gives us (3.3.72). □

Next, we have an estimate that compares L_μ^6 and L_μ^8 norms to L_μ^4 norms.

Lemma 12. *Let $\Sigma^2 \subset \mathbb{R}^3$ be a critical hyper-surface to the gaussian isoperimetric problem with $H = \frac{\langle x, N \rangle}{2} + C$. Also let $R \geq 0$ and $\phi \in C_0^1(\Sigma \cap B_R(0))$. There exists a constant D such that we have the estimates*

$$\int_{B_R(0)} \phi^4 u^6 dA_\mu \leq D \mathcal{O}_C^2 \|u\|_{4,R}^6 e^{\frac{R^2}{8}} \left(\mathcal{O}_\phi \mathcal{O}_R + e^{\frac{R^2}{8}} \|u\|_{4,R}^2 \right), \quad (3.3.80)$$

$$\int_{B_R(0)} \phi^4 u^8 dA_\mu \leq D \mathcal{O}_C^4 e^{\frac{R^2}{4}} \|u\|_{4,R}^8 \left(\mathcal{O}_\phi^2 \mathcal{O}_R^2 + e^{\frac{R^2}{4}} \|u\|_{4,R}^4 \right). \quad (3.3.81)$$

Proof. Using Holder's inequality we have that

$$\int_{B_R(0)} \phi^4 u^6 dA_\mu \leq \left(\int_{B_R(0)} \phi^4 u^4 dA_\mu \right)^{\frac{1}{2}} \left(\int_{B_R(0)} \phi^4 u^8 dA_\mu \right)^{\frac{1}{2}}, \quad (3.3.82)$$

$$= \|\phi u\|_4^2 \|\phi u^2\|_4^2. \quad (3.3.83)$$

Now, we use (3.3.55) to get

$$\|\phi u^2\|_4^4 \leq D_1 \mathcal{O}_C^2 e^{\frac{R^2}{4}} \|\phi u^2\|_2^2 (\mathcal{O}_\phi^2 \mathcal{O}_R^2 \|u^2\|_{2,R}^2 + \|\phi u^3\|_2^2). \quad (3.3.84)$$

Using Holder's inequality, we have that $\|\phi u^3\|_2^2 \leq \|u\|_{4,R}^2 \|\phi u^2\|_4^2$. From a Cauchy ϵ -inequality of the form $ab \leq \epsilon \frac{a^2}{2} + \frac{b^2}{2\epsilon}$ we obtain that

$$D_1 \mathcal{O}_C^2 e^{\frac{R^2}{4}} \|\phi u^2\|_2^2 \|u\|_{4,R}^2 \|\phi u^2\|_4^2 \leq D_2 \mathcal{O}_C^4 e^{\frac{R^2}{2}} \|\phi u^2\|_2^4 \|u\|_{4,R}^4 + \frac{1}{2} \|\phi u^2\|_4^4. \quad (3.3.85)$$

Therefore, from (3.3.84) we get that

$$\|\phi u^2\|_4^4 \leq D_3 \mathcal{O}_C^4 \|u\|_{4,R}^8 e^{\frac{R^2}{4}} \left(\mathcal{O}_\phi^2 \mathcal{O}_R^2 + e^{\frac{R^2}{4}} \|u\|_{4,R}^4 \right). \quad (3.3.86)$$

This gives us (3.3.81).

Then, using (3.3.83), we have that

$$\int_{B_R(0)} \phi^4 u^6 dA_\mu \leq D_4 \mathcal{O}_C^2 \|u\|_{4,R}^6 e^{\frac{R^2}{8}} \left(\mathcal{O}_\phi \mathcal{O}_R + e^{\frac{R^2}{8}} \|u\|_{4,R}^2 \right). \quad (3.3.87)$$

□

Now we may combine Proposition 2 and Proposition 3 to obtain pointwise curvature estimates for $|A|$.

Theorem 3. *Let $\Sigma \subset B_R(0) \subset \mathbb{R}^3$ be a smooth stable critical hyper-surface of the gaussian isoperimetric problem with $H = \frac{1}{2} \langle x, N \rangle + C$ and $\partial \Sigma \subset \partial B_R(0)$. Consider any $0 < \theta < 1$. We have that there exists a constant D (independent of Σ , θ , and R) such that if $\theta(1-\theta)R > D\mathcal{O}_C$ and $A_E(\Sigma) < D e^{-D\mathcal{O}_C^2} \mathcal{O}_C^n (1-\theta)^2 R^2 e^{\frac{(1-\theta)R^2}{16}}$, then*

$$\sup_{B_{\theta R}(0)} |A| \leq D \mathcal{O}_C^{\frac{1}{2}} e^{D_4 \mathcal{O}_C^2} (1-\theta)^{-4} A_E (1 + A_E) \mathcal{O}_R^{\frac{1}{4}} e^{-\frac{1}{32}(1-\theta^2)R^2}. \quad (3.3.88)$$

Proof. First, we restrict ourselves to $R \geq D_1^{-1}$ so that $R^{-1} \leq D_1 \mathcal{O}_R^{-1}$.

Consider $\theta_1 = \frac{\theta+1}{2}$, and similarly recursively define $\theta_{i+1} = \frac{\theta_i+1}{2}$. Note that $\theta \leq \theta_1 \leq \theta_2 \leq \dots \leq 1$.

We also have that $1 - \theta_i = 2^{-i}(1 - \theta)$, and $\theta_{i+1} - \theta_i = 2^{-i-1}(1 - \theta)$. See Figure 3.2 below.

We first apply Proposition 3 to the balls $B_{\theta R}(0) \subset B_{\theta_1 R}(0)$. We have that

$$\sup_{B_{\theta R}(0)} |A| \leq D_2 \mathcal{O}_R \mathcal{O}_C^{\frac{3}{2}} (1-\theta)^{-1} e^{\frac{1}{8}(\frac{1}{3}\theta^2 + \frac{2}{3}\theta_1^2)R^2} (1 + e^{\frac{\theta_1^2 R^2}{16}} \|u\|_{6,\theta_1 R}^{\frac{3}{2}}) \|u\|_{2,\theta_1 R} \quad (3.3.89)$$

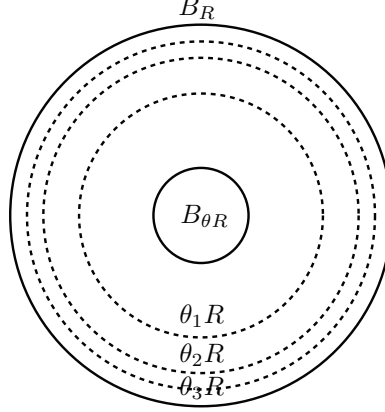


Figure 3.2: Set up of $\theta_i R$

We need to estimate $\|u\|_{6,\theta_1 R}^{\frac{3}{2}}$. From Lemma 12, we have that

$$\|u\|_{6,\theta_1 R}^{\frac{3}{2}} \leq D_3 \mathcal{O}_C^{\frac{1}{2}} \|u\|_{4,\theta_2 R}^{\frac{3}{2}} e^{\frac{\theta_2 R^2}{32}} \left(\left(1 + \frac{1}{1-\theta}\right)^{\frac{1}{4}} \mathcal{O}_R^{\frac{1}{4}} + e^{\frac{\theta_2 R^2}{32}} \|u\|_{4,\theta_2 R}^{\frac{1}{2}} \right). \quad (3.3.90)$$

Now, we want to use Lemma 11 to estimate $\|u\|_{4,\theta_2 R}^{\frac{1}{2}}$. To do so, we need to guarantee the sufficient condition,

$$D_4 \mathcal{O}_C^2 e^{\frac{\theta_3^2 R^2}{4}} \int_{B_{\theta_3 R}(0)} u^2 dA_\mu \leq 1, \quad (3.3.91)$$

of Lemma 11. To guarantee (3.3.91) is true, we want to estimate $\|u\|_{2,\theta_4 R}^2$ using Proposition 2. To apply Proposition 2, it is sufficient that $(1 - \theta_4)\theta_4 R > D_5 \mathcal{O}_C$ and $A_E(\Sigma \cap B_R(0)) < D_5^{-1} e^{-D_5 \mathcal{O}_C^2} \mathcal{O}_C^2 (1 - \theta)^2 R^2 e^{\frac{(\theta_4^2 - \theta_3^2) R^2}{16}}$. This gives us conditions to impose on Σ .

So, from Proposition 2, we have that (3.3.91) becomes

$$D_6 \mathcal{O}_C e^{\frac{\theta_3^2 R^2}{4}} \int_{B_{\theta_3 R}(0)} u^2 dA_\mu \leq \mathcal{O}_C e^{\frac{\theta_3^2 R^2}{4}} \int_{B_{\theta_4 R}(0)} u^2 dA_\mu, \quad (3.3.92)$$

$$\leq D_6 e^{D_6 \mathcal{O}_C^2} \mathcal{O}_C^{-1} \frac{A_E(\Sigma \cap B_{\theta_3 R}(0))^2}{(1 - \theta)^2} \theta_4^{-2} R^{-2} e^{\frac{(\theta_3^2 - \theta_4^2) R^2}{4}}, \quad (3.3.93)$$

$$\leq D_7 e^{D_7 \mathcal{O}_C^2} \mathcal{O}_C (1 - \theta)^2 R^2 e^{\frac{(\theta_3^2 - \theta_4^2) R^2}{8}}, \quad (3.3.94)$$

$$\leq 1. \quad (3.3.95)$$

For the last inequality, we used that $\theta_3^2 - \theta_4^2 \leq -\frac{29}{16}(1 - \theta)^2$ and the conditions we impose on Σ with a choice of larger constant D_7 . Note that we have used Proposition 2 to estimate $\|u\|_{2,\theta_4 R}^2$ instead of $\|u\|_{2,\theta_3 R}^2$ so that we get a negative exponential in (3.3.94).

So, now we may apply Lemma 11 to get $\|u\|_{4,\theta_2 R}^{\frac{1}{2}} \leq D_8(1-\theta)^{-\frac{1}{4}} \mathcal{O}_R^{\frac{1}{4}} \|u\|_{2,\theta_3 R}^{\frac{1}{4}}$. Hence, we have that (3.3.90) becomes

$$\|u\|_{6,\theta_1 R}^{\frac{3}{2}} \leq D_9 \mathcal{O}_C^{\frac{1}{2}} (1-\theta)^{-1} \mathcal{O}_R e^{\frac{\theta_2^2 R^2}{32}} \|u\|_{2,\theta_3 R}^{\frac{3}{4}} \left(1 + \|u\|_{2,\theta_3 R}^{\frac{1}{4}} e^{\frac{\theta_2^2 R^2}{32}}\right). \quad (3.3.96)$$

Thus far, the conditions we have imposed on Σ are sufficient to allow us to apply Proposition 2 to get the estimate $\|u\|_{2,\theta_3 R}^{\frac{1}{4}} \leq D_{10} e^{D_{10} \mathcal{O}_C^2} \mathcal{O}_C^{-\frac{1}{4}} (1-\theta)^{-\frac{1}{4}} A_E^{\frac{1}{4}} R^{-\frac{1}{4}} e^{-\frac{\theta_3^2 R^2}{32}}$.

So, using (3.3.96), we then have that

$$\|u\|_{6,\theta_1 R}^{\frac{3}{2}} \leq D_{11} e^{D_{11} \mathcal{O}_C^2} (1-\theta)^{-2} \mathcal{O}_R^{\frac{1}{4}} e^{-\frac{\theta_2^2 R^2}{16}} A_E^{\frac{3}{4}} \left(1 + A_E^{\frac{1}{4}}\right). \quad (3.3.97)$$

So,

$$e^{\frac{\theta_1^2 R^2}{16}} \|u\|_{6,\theta_1 R}^{\frac{3}{2}} \leq D_{12} e^{D_{12} \mathcal{O}_C^2} (1-\theta)^{-2} \mathcal{O}_R^{\frac{1}{4}} A_E^{\frac{3}{4}} \left(1 + A_E^{\frac{1}{4}}\right). \quad (3.3.98)$$

Therefore, (3.3.89) becomes

$$\sup_{B_{\theta R}(0)} |A| \leq D_{13} \mathcal{O}_C^{\frac{3}{2}} e^{D_{13} \mathcal{O}_C^2} (1-\theta)^{-3} \mathcal{O}_R^{\frac{5}{4}} (1 + A_E) e^{\frac{1}{8}(\frac{1}{3}\theta^2 + \frac{2}{3}\theta_1^2)R^2} \|u\|_{2,\theta_1 R}. \quad (3.3.99)$$

Now, note that $\|u\|_{2,\theta_1 R} \leq \|u\|_{2,\theta_3 R}$, and so we have that

$$\sup_{B_{\theta R}(0)} |A| \leq D_{14} \mathcal{O}_C^{\frac{1}{2}} e^{D_{14} \mathcal{O}_C^2} (1-\theta)^{-4} A_E (1 + A_E) \mathcal{O}_R^{\frac{1}{4}} e^{\frac{1}{8}(\frac{1}{3}\theta^2 + \frac{2}{3}\theta_1^2 - \theta_3^2)R^2}, \quad (3.3.100)$$

$$\leq D_{14} \mathcal{O}_C^{\frac{1}{2}} e^{D_{14} \mathcal{O}_C^2} (1-\theta)^{-4} A_E (1 + A_E) \mathcal{O}_R^{\frac{1}{4}} e^{\frac{1}{24}(\theta^2 - \theta_3^2)R^2}. \quad (3.3.101)$$

Now, note that $\theta_4(1-\theta_4) > \frac{1}{16}\theta(1-\theta)$, and that $e^{\frac{(\theta_4^2 - \theta_3^2)R^2}{16}} > e^{\frac{(1-\theta)R^2}{16}}$. So therefore, if we meet the sufficient conditions of the theorem, then we meet the sufficient conditions needed to apply the lemmas that we have used throughout the course of the proof. Also, for simplification, note that $\frac{1}{24}(\theta^2 - \theta_3^2) \leq -\frac{1}{32}(1-\theta^2)$.

□

4

Gaussian Harmonic One Forms

Now, we consider results by the author [26] concerning the consequences of the construction of Gaussian Harmonic one-forms on a self-shrinker $M^2 \subset \mathbb{R}^n$. Gaussian Harmonic One-Forms are closed one-forms satisfying the Euler Lagrange equation for minimizing the weighted norm $\int_{\Sigma} |\omega|^2 dA_{\mu}$ in a cohomology class.

Definition 10. A one-form ω is a **Gaussian Harmonic One-Form** if

- ω is closed ($\nabla\omega$ is symmetric; i.e. $\nabla_i\omega_j = \nabla_j\omega_i$).
- The trace of $\nabla\omega$ satisfies $\nabla_i\omega_i = \frac{1}{2}\omega(x^T)$ where x is the position vector in euclidean space.

For $M^2 \subset \mathbb{R}^n$, we will use $H_{\mu}(M)$ to denote the space of $L_{\mu}^2(M)$ gaussian harmonic one-forms on M .

Much like the case of harmonic one-forms on Riemann surfaces, we have that the genus of a surface Σ gives us a lower bound on the dimension of the L_{μ}^2 Gaussian Harmonic One-forms that may be constructed.

Lemma 13. Let $M^2 \subset \mathbb{R}^n$ be a sub-manifold with polynomial volume growth and genus g . Then $\dim H_{\mu}(M) \geq g$.

Proof. The proof is similar to that of the euclidean case [15, 23]. First consider g disjoint non-separating Jordan curves $\gamma_i \subset M$. There exists dual forms η_i to each γ_i such that their supports are in small neighborhoods of each γ_i , and for any one-form ω we have that $\int_{\gamma_i} \omega = \int_{\Sigma} \eta_i \wedge \omega$. Note that these integrals are just integrals of differential forms and do not involve dA_{μ} .

Since the γ_i are non-separating, we may find Jordan curves α_i such that α_i intersects γ_i exactly once, and we also have that α_i does not intersect γ_j for $i \neq j$. One may use the dual forms τ_i to α_i

to show that the η_i are linearly independent. Furthermore, for any $f \in C_0^\infty(M)$ we have that $\eta_i + df$ are linearly independent.

Now, we consider the corresponding weak versions of the definition for ω being a Gaussian Harmonic One-Form. Let $A \subset L_\mu^2(M)$ be the closed subspace $A = \overline{\text{Span}\{df : f \in C_0^\infty(M)\}}$ (A^\perp is the space of forms that weakly satisfy $\nabla_i \omega_i = \frac{1}{2} \omega(x^T)$), and let $B \subset L_\mu^2(M)$ be the closed subspace $B = \overline{\text{Span}\{\delta\psi + \frac{1}{2} \text{Inn}(x^T)\psi : \psi \text{ is a } C_0^\infty \text{ 2-form on } M\}}$ (B is the space of weakly closed forms). Note that $\delta + \frac{1}{2} \text{Inn}(x^T)$ is the adjoint of d for $L_\mu^2(M)$. Hence, we see that A and B are orthogonal in $L_\mu^2(M)$. Let $H_0 = (A \oplus B)^\perp$. We see that H_0 is the space of $L_\omega^2(M)$ forms that are weakly Gaussian Harmonic.

Since, the η_i are closed, we see that $\eta_i \in H_0 \oplus A$. Let ω_i be the projection of η_i onto H_0 . Since we have for any $f \in C_0^\infty(M)$ that $\int_M \tau_i \wedge (\eta_j + df) = \int_{\alpha_i} \eta_j + df = \delta_{ij}$, we see that the ω_i are linearly independent. Note that since the $\text{supp } \tau_i$ is compact, we have that the above relation still holds in the $L_\mu^2(\Sigma)$ limit.

Therefore, $\text{Span}\{\omega_i\}$ is a g -dimensional space of weakly Gaussian Harmonic One-Forms. So, once we show that the ω_i are actually C^∞ , then we are done.

Since $\omega_i \in H_0$, we have for any C_0^∞ function f and C_0^∞ two-form ψ that

$$0 = \int_M \langle \omega_i, df \rangle dA_\mu, \quad (4.0.1)$$

$$0 = \int_M \langle \omega_i, \delta\psi + \frac{1}{2} \text{Inn}(x^T)\psi \rangle dA_\mu. \quad (4.0.2)$$

$$(4.0.3)$$

Plugging in $f \rightarrow f e^{\frac{|x|^2}{4}}$ in (4.0.1) and $\psi \rightarrow \psi e^{\frac{|x|^2}{4}}$ into (4.0.3), we may rewrite both equations as

$$0 = \int_M \langle \omega_i, df + \frac{1}{4} f d|x|^2 \rangle dA. \quad (4.0.4)$$

$$0 = \int_M \langle \omega_i, \delta\psi \rangle dA. \quad (4.0.5)$$

Now, we use local conformal flat coordinates (u, v) . Note that (4.0.4) and (4.0.5) are conformally invariant. Let $\omega_i = p du + q dv$. For any $\phi \in C_0^\infty(M)$ supported on the coordinate chart, let $f = \phi_v$

and $\psi = \phi_u du \wedge dv$. Then (4.0.4) and (4.0.5) become

$$0 = \int_M \left(p\phi_{uv} + q\phi_{vv} + \frac{\partial_u |x|^2}{4} p\phi_v + \frac{\partial_v |x|^2}{4} q\phi_v \right) du dv, \quad (4.0.6)$$

$$0 = \int_M (q\phi_{uu} - p\phi_{uv}) du dv. \quad (4.0.7)$$

Adding these two equations we obtain

$$0 = \int_M \left(q\Delta_{uv}\phi + \frac{\partial_u |x|^2}{4} p\phi_v + \frac{\partial_v |x|^2}{4} q\phi_v \right) du dv \quad (4.0.8)$$

Now, using $f = \phi_u$ and $\psi = \phi_v du \wedge dv$, we get

$$0 = \int_M \left(p\Delta_{uv}\phi + \frac{\partial_u |x|^2}{4} p\phi_u + \frac{\partial_v |x|^2}{4} q\phi_u \right) du dv. \quad (4.0.9)$$

Equations (4.0.8) and (4.0.9) give us a weakly linear elliptic system for p and q . From standard linear elliptic system theory, we have that p and q must be C^∞ on M [16, 13, 30].

□

Now, we consider some identities for Gaussian Harmonic Forms.

Lemma 14. *Let ω be a Gaussian Harmonic Form on a self-shrinker $M^2 \subset \mathbb{R}^n$ with dual vector field $W \in TM$. For any $v \in T_p M$, we have that*

$$(\mathcal{L}^\Sigma \omega)(v) = \frac{1}{2} \omega(v) - A^\alpha(W, e_i) A^\alpha(e_i, v). \quad (4.0.10)$$

Proof. We first use a Weitzenböck formula $\Delta^M \omega = -(d\delta + \delta d)\omega + K\omega$. Here, $-(d\delta + \delta d)$ is the Hodge Laplacian on differential forms while K is the scalar curvature of M [24]. We have that $\delta\omega = -\text{Div } \omega = -\frac{1}{2}\omega(x^T)$, and $d\omega = 0$. So the Hodge Laplacian $-(d\delta + \delta d)\omega = \frac{1}{2}d[\omega(x^T)] = \frac{1}{2}\nabla^\Sigma[\omega(x^T)]$. So, from a Leibniz rule, we get that

$$(\Delta^M \omega)(v) = K\omega(v) + \frac{1}{2}\omega(\nabla_v^M x^T) + \frac{1}{2}\nabla^M \omega(x^T, v). \quad (4.0.11)$$

For a self-shrinker, we have that $\nabla_v^M x^T = v - 2A^H(v, e_i)e_i$. So, we get

$$(\mathcal{L}^M \omega)(v) = K\omega(v) + \frac{1}{2}\omega(v) - A^H(v, W). \quad (4.0.12)$$

Now, let $\{\eta_\alpha\}$ be an orthonormal frame for the normal bundle. Let $\kappa_{\alpha i}$ be the eigenvalues of A^{η_α} with orthonormal eigenvectors $e_{\alpha i}$. Note that $A^H = H_\alpha A^\alpha = (\kappa_{\alpha 1} + \kappa_{\alpha 2})A^\alpha$. Here we summed over different orthonormal frames in $T_p M$ for each α . Now, let $W_{\alpha i}$ be the components of W in the frame $\{e_{\alpha i}\}$ for fixed α . Using the Gauss equation, we have that

$$KW - \sum_i A^H(W, e_i)e_i = \sum_{\alpha, i} [\kappa_{\alpha 1}\kappa_{\alpha 2}W_{\alpha i}e_{\alpha i} - (\kappa_{\alpha 1} + \kappa_{\alpha 2})\kappa_{\alpha i}W_{\alpha i}e_{\alpha i}], \quad (4.0.13)$$

$$= \sum_{\alpha, i} -\kappa_{\alpha i}^2 W_{\alpha i}e_{\alpha i}, \quad (4.0.14)$$

$$= -A^\alpha(W, e_i)A^\alpha(e_i, e_j)e_j. \quad (4.0.15)$$

Note, that in (4.0.15), we are able to turn the sum over $e_{\alpha i}$ into a sum over any frame e_i , because the trace of a tensor is independent of the choice of orthonormal frame. From here, we have (4.0.10) \square

Now, we consider the differential operator \mathcal{L}^E acting on vector fields on M viewed as vector fields in \mathbb{R}^n . During our calculations, we will often need to make use of vectors and coordinates in euclidean space. We will use indices such as a, b, \dots to denote euclidean vectors or components. For example, the vectors $\{\partial_a\}$ are an orthonormal basis in \mathbb{R}^n , and for any vector v we may write $v = v^a \partial_a$.

Lemma 15. *Let ω be a Gaussian Harmonic One-Form on a self-shrinker $M^2 \subset \mathbb{R}^n$. Let W be the vector field dual to ω . We have that*

$$\mathcal{L}^E W = -2\langle \nabla^M \omega, A^\alpha \rangle \eta_\alpha + \frac{1}{2}W - 2A^\beta(W, e_j)A^\beta(e_j, e_k)e_k. \quad (4.0.16)$$

Proof. We note that $W = \langle \omega, dx^b \rangle \partial_b$. So therefore we have that

$$\mathcal{L}^E W = \langle \mathcal{L}^M \omega, dx^b \rangle \partial_b + 2\langle \nabla^M \omega, \nabla^M dx^b \rangle \partial_b + \langle \omega, \mathcal{L}^M dx^b \rangle \partial_b. \quad (4.0.17)$$

We need to compute the three terms on the right hand side of (4.0.17). First, we examine the quantity $\langle \nabla^M \omega, \nabla^M dx^b \rangle \partial_b$. We see that $\nabla^M \partial_a^T = -\langle \partial_b, \eta_\alpha \rangle \nabla^M \eta_\alpha$. Hence, we have that $\nabla^M dx^b = -\eta_\alpha^b A^\alpha$. So, we get

$$\langle \nabla^M \omega, \nabla^M dx^b \rangle \partial_b = -\langle \nabla^M \omega, A^\alpha \rangle \eta_\alpha. \quad (4.0.18)$$

Next, we compute $\langle \omega, \mathcal{L}^M dx^b \rangle \partial_b$. We fix a point $p \in M$. Then, we use a geodesic tangential frame $\{e_i\}$ and normal frame $\{\eta_j\}$ such that $\nabla^M e_i(p) = 0$ and $\nabla^N \eta_j(p) = 0$. From the Codazzi Equation,

we have that

$$(\Delta^M dx^b)(e_i) = -A^\alpha(\partial_b^T, e_j)A^\alpha(e_j, e_i) - (\nabla_j^N H)^b, \quad (4.0.19)$$

$$= -A^\alpha(\partial_b^T, e_j)A^\alpha(e_j, e_i) - \frac{\eta_\alpha^b}{2}A^\alpha(x^T, W). \quad (4.0.20)$$

So, we see that we have that

$$\langle \omega, \mathcal{L}^M dx^b \rangle \partial_b = -A^\alpha(W, e_j)A^\alpha(e_j, e_k)e_k. \quad (4.0.21)$$

So, using equations (4.0.10), (4.0.18), and (4.0.21) we then get (4.0.16).

□

Now we prove a result that is like a rigidity theorem for the genus of M .

Theorem 4. *If $M^2 \subset \mathbb{R}^N$ is an orientable self-shrinker of polynomial volume growth with genus $g \geq 1$, then*

$$\sup_{x \in M, v \in T_x M, |v|=1} A^\alpha(v, i)A^\alpha(i, v) \geq \frac{1}{2}. \quad (4.0.22)$$

Proof. From Lemma 13, we have that there are g linearly independent gaussian harmonic forms in $L_\mu^2(M)$.

Consider any Gaussian Harmonic Form $\omega \in L_\mu^2(M)$ with dual (with respect to the euclidean metric on M) vector field W . Consider any $\phi \in C_0^\infty(M)$. From a calculation similar to the proof of (3.2.6) in Lemma 4, we have that

$$0 \leq \int_M |\nabla^M(\phi W)|^2 dA_\mu, \quad (4.0.23)$$

$$= \int_M |W|^2 |\nabla \phi|^2 dA_\mu - \int_M \langle W, \mathcal{L}^M W \rangle dA_\mu. \quad (4.0.24)$$

Then, using (4.0.10), we get that

$$0 \leq \int_M |W|^2 |\nabla \phi|^2 dA_\mu + \int_M \phi^2 A^\alpha(W, i)A^\alpha(i, W) dA_\mu - \frac{1}{2} \int_M \phi^2 |W|^2 dA_\mu. \quad (4.0.25)$$

For clarity of notation, let $S = \sup_{x \in M, v \in T_x M, |v|=1} A^\alpha(v, i)A^\alpha(i, v)$. Since $W \in L_\mu^2(M)$, we may use

standard cut-off functions exhausting M and with $|\nabla^M \phi| \leq 1$ to show that (4.0.25) gives us that

$$0 \leq \left(S - \frac{1}{2}\right) \int_M |W|^2. \quad (4.0.26)$$

As we remarked earlier, since the genus $g \geq 1$, we have that the space of $L_\mu^2(M)$ gaussian harmonic one-forms is non-zero. Therefore, we get that $S \geq \frac{1}{2}$.

□

Now, using methods of Ros [33] and Urbano [37], we show that if the principal curvatures of M are not too far from each other in absolute value, then we have a lower bound for the index of L acting on C_0^∞ functions on M .

Theorem 5. *Let $M^2 \subset \mathbb{R}^n$ be an orientable self-shrinker of polynomial volume growth. If*

$\sup_{p \in M} \inf_{\{\eta_\alpha\} \text{ orth.}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| \leq \delta < 1$, then the index of L acting on scalar functions on M has a lower bound given by

$$\text{Index}_M(L) \geq \frac{g}{n}. \quad (4.0.27)$$

Proof. We may assume $\text{Index}_M(L) = J < \infty$. From Fischer-Colbrie [17], we know that there exist $L_\mu^2(M)$ functions ψ_1, \dots, ψ_J such that if $f \in C_0^\infty(M)$ and $\int_M f \psi_i dA_\mu = 0$ for all i , then we have that $-\int_M f L f dA_\mu \geq 0$.

For now, let us consider the case that $g < \infty$. From Lemma 13, we have that there are g linearly independent $L_\mu^2(M)$ gaussian harmonic one-forms ω_i on M . Let W_i be the dual vector field of ω_i with respect to the euclidean metric, and let $V = \text{Span}\{W_i\}$. Let $W \in V$. From a calculation similar to that of (3.2.6), we have that

$$-\int_M \langle \phi W, L^E(\phi W) \rangle dA_\mu = \int_M |W|^2 |\nabla \phi|^2 - \int_M \phi^2 \langle W, L^E W \rangle dA_\mu. \quad (4.0.28)$$

So, then (4.0.16) gives us that

$$\begin{aligned} \int_M \phi^2 \langle W, L^E W \rangle dA_\mu &= \int_M \phi^2 |W|^2 dA_\mu + \int_M \phi^2 |A|^2 |W|^2 dA_\mu \\ &\quad - 2 \int_M \phi^2 A^\alpha(W, e_i) A^\alpha(e_i, W) dA_\mu. \end{aligned} \quad (4.0.29)$$

Now, note that

$$2A^\alpha(W, e_i)A^\alpha(e_i, W) - |A|^2|W|^2 = \sum_\alpha 2 \left(\sum_i \kappa_{\alpha i}^2 W_{\alpha i}^2 \right) - (\kappa_{\alpha 1}^2 + \kappa_{\alpha 2}^2)(W_{\alpha 1}^2 + W_{\alpha 2}^2), \quad (4.0.30)$$

$$= \sum_\alpha \sum_{i \neq j} (\kappa_{\alpha i}^2 - \kappa_{\alpha j}^2) W_{\alpha i}^2, \quad (4.0.31)$$

$$\leq \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| |W|^2. \quad (4.0.32)$$

So, (4.0.29) becomes

$$\int_M \phi^2 \langle W, L^E W \rangle dA_\mu \geq \int_M \phi^2 |W|^2 dA_\mu - \int_M \phi^2 |W|^2 \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| dA_\mu. \quad (4.0.33)$$

Now, at every point $x \in M$, one is allowed to choose to sum over any frame η_α for the normal bundle. So, one gets

$$\int_M \phi^2 \langle W, L^E W \rangle dA_\mu \geq \int_M \phi^2 |W|^2 dA_\mu - \int_M \phi^2 |W|^2 \inf_{\{\eta_\alpha\}} \sum_\alpha |\kappa_{\alpha 1}^2 - \kappa_{\alpha 2}^2| dA_\mu, \quad (4.0.34)$$

$$\geq (1 - \delta) \int_M \phi^2 |W|^2 dA_\mu. \quad (4.0.35)$$

Since $\text{Dim}(V) < \infty$ and $V \subset L_\mu^2(M)$, an argument similar to that given in Proposition 1 shows that there exists a cut-off function $\phi \in C_0^\infty(M)$ with $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq 1$ such that $\text{Dim}(\phi V) = \text{Dim}(V)$. Furthermore, for any $W \in V$ we have that $\int_M \phi^2 |W|^2 dA_\mu \geq \frac{2}{3} \int_M |W|^2 dA_\mu$, and $\int_M |\nabla \phi|^2 |W|^2 dA_\mu \leq \frac{1-\delta}{3} \int_M |W|^2 dA_\mu$. Then using (4.0.35) for this cut-off function $\phi \in C_0^\infty(\Sigma)$, we have that

$$- \int_M \langle \phi W, L^E \phi W \rangle dA_\mu \leq (\delta - 1) \int_M \phi^2 |W|^2 dA_\mu + \int_M |\nabla \phi|^2 |W|^2 dA_\mu, \quad (4.0.36)$$

$$\leq \frac{\delta - 1}{3} \int_M |W|^2 dA_\mu. \quad (4.0.37)$$

So, $-\int_M \langle \phi W, L^E \phi W \rangle dA_\mu < 0$ if $W \neq 0$.

Now, consider the linear map $F : \phi V \rightarrow \mathbb{R}^{n_J}$ defined by

$$F(\phi W) = \left(\int_M \phi \psi_1 W dA_\mu, \dots, \int_M \phi \psi_J W dA_\mu \right). \quad (4.0.38)$$

So, if $\phi W \in \text{Ker } F$, then we have that $\int_M \langle \phi W, L^E \phi W \rangle dA_\mu \geq 0$. Therefore, $\phi W \equiv 0$. So, $g = \text{Dim}(V) = \text{Dim}(\phi V) \leq nJ$, and the theorem follows for $g < \infty$.

For $g = \infty$, we need only take subspaces of finite dimension and work as above to get that $J \geq \frac{m}{n}$ for all m . This gives $J = \infty$.

□

Appendix

Consider $M \subset N$ and a compact normal variation $F(x, t)$ of M . For completeness, we first prove a standard formula for $\bar{\nabla}_{F_t} H|_{t=0}$ [9].

Lemma 16.

$$\bar{\nabla}_{F_t} H|_{t=0} = -\nabla_{ii}^{N,2} F_t - \langle A_{ij}, F_t \rangle A_{ij} - [\bar{R}(F_t, F_i) F_i]^N - \langle A_{ij}, \bar{\nabla}_{F_i} F_t \rangle F_i. \quad (\text{A.0.39})$$

Proof. Consider a point $p \in M$ and geodesic coordinates on M centered at p . Note that $\bar{\nabla}_{F_i} F_j = \nabla_{F_i}^N F_j$ when $t = 0$. Also, take note that for all time t we have $\bar{\nabla}_{F_i} F_j = \bar{\nabla}_{F_j} F_i$ and $\bar{\nabla}_{F_t} F_i = \bar{\nabla}_{F_i} F_t$. Let $g_{ij}(t)$ be the metric on $F(\cdot, t)$.

Now, use that $H = -g^{ij} \nabla_{F_i}^N F_j = -g^{ij} (\bar{\nabla}_{F_i} F_j - \nabla_{F_i} F_j)$.

Using $[F_t, F_i] = 0$, note that $\bar{\nabla}_{F_i} (g^{ij}) = -2g^{il} g^{jm} \langle A_{lm}, F_t \rangle = -2\langle A_{ij}, F_t \rangle$ at p . We also get

$$\bar{\nabla}_{F_t} (\bar{\nabla}_{F_i} F_i) = \bar{R}(F_t, F_i) F_i + \bar{\nabla}_{F_i} (\bar{\nabla}_{F_t} F_i), \quad (\text{A.0.40})$$

$$= \bar{R}(F_t, F_i) F_i + \bar{\nabla}_{F_i} (\bar{\nabla}_{F_t} F_t). \quad (\text{A.0.41})$$

Again using normal coordinates, we have at p that

$$-\bar{\nabla}_{F_t} (\nabla_{F_i} F_i) = -\bar{\nabla}_{F_t} (\langle \bar{\nabla}_{F_i} F_i, F_j \rangle g^{jk} F_k), \quad (\text{A.0.42})$$

$$= -\langle \bar{\nabla}_{F_t} (\bar{\nabla}_{F_i} F_i), F_j \rangle g^{jk} F_k - \langle \bar{\nabla}_{F_i} F_i, \bar{\nabla}_{F_t} F_j \rangle g^{jk} F_k, \quad (\text{A.0.43})$$

$$= -[\bar{R}(F_t, F_i) F_i]^T - [\bar{\nabla}_{F_i} (\bar{\nabla}_{F_t} F_t)]^T + \langle A_{ii}, \bar{\nabla}_{F_j} F_t \rangle g^{jk} F_k. \quad (\text{A.0.44})$$

$$(\text{A.0.45})$$

Therefore, at p we get

$$\bar{\nabla}_{F_t} H = -2\langle A_{ij}, F_t \rangle A_{ij} - [\bar{R}(F_t, F_i) F_i]^N - \nabla_{F_i}^N (\bar{\nabla}_{F_i} F_t) - \langle A_{ii}, \bar{\nabla}_{F_j} F_t \rangle F_j. \quad (\text{A.0.46})$$

Note that

$$\nabla_{F_i}^N (\bar{\nabla}_{F_i} F_t) = \nabla_{ii}^{N,2} F_t + \nabla_{F_i}^N (\langle \bar{\nabla}_{F_i} F_t, F_j \rangle F_j), \quad (\text{A.0.47})$$

$$= \nabla_{ii}^{N,2} F_t + \langle \bar{\nabla}_{F_i} F_t, F_j \rangle \nabla_{F_i}^N F_j, \quad (\text{A.0.48})$$

$$= \nabla_{ii}^{N,2} F_t - \langle A_{ij}, F_t \rangle A_{ij}. \quad (\text{A.0.49})$$

Therefore, at p

$$\bar{\nabla}_{F_t} H = -\nabla_{ii}^{N,2} F_t - \langle A_{ij}, F_t \rangle A_{ij} - [\bar{R}(F_t, F_i) F_i]^N - \langle A_{ij}, \bar{\nabla}_{F_i} F_t \rangle F_i, \quad (\text{A.0.50})$$

and the lemma follows. \square

Now we may prove Lemma 1.

Proof of Lemma 1. Note that for all time t we have

$$\frac{\partial}{\partial t} A_f(U_t) = \int_{U_t} e^f \langle \bar{\nabla} f + H, F_t \rangle dA. \quad (\text{A.0.51})$$

Since M is f -minimal, we have that

$$\frac{\partial^2}{\partial t^2} A_f(U_t) \Big|_{t=0} = \int_U e^f \langle \bar{\nabla}_{F_t} (\bar{\nabla} f + H), F_t \rangle dA. \quad (\text{A.0.52})$$

Since F is a normal variation, we have from (A.0.39) that

$$\int_U e^f \langle \bar{\nabla}_{F_t} H, F_t \rangle dV = - \int_U e^f \left\langle \nabla_{ii}^{N,2} F_t + \langle A_{ij}, F_t \rangle A_{ij}, F_t \right\rangle + e^f \overline{\text{Ric}}(F_t, F_t) dA. \quad (\text{A.0.53})$$

Also, note that $\langle \bar{\nabla}_{F_t} \bar{\nabla} f, F_t \rangle = \overline{\text{Hess}_f}(F_t, F_t)$. Therefore, we get the lemma. \square

For completeness, we prove a common formula for $\frac{\partial}{\partial t} N|_{t=0}$ using the found in McGonagle-Ross [27].

Lemma 17. *For any normal variation of a hyper-surface $\Sigma \subset \mathbb{R}^{n+1}$ with $u \equiv F_t|_{t=0}$, we have that*

$$\left. \frac{\partial}{\partial t} N \right|_{t=0} = -\nabla u. \quad (\text{A.0.54})$$

Proof. Fix an orientation on \mathbb{R}^{n+1} such that the $n+1$ -form $N \wedge \left(\bigwedge_i F_i \right)$ is positive. We also fix a choice of sign for the Hodge star $*$ operator such that $*\omega \wedge \omega$ is positive for this orientation. Then, we have that

$$N = \frac{* \bigwedge_i F_i}{\left| \bigwedge_i F_i \right|}. \quad (\text{A.0.55})$$

Note that

$$\partial_t \left| \bigwedge_i F_i \right| = \left| \bigwedge_i F_i \right| g^{jk} \langle \partial_j F_t, \partial_k \rangle, \quad (\text{A.0.56})$$

$$= \left| \bigwedge_i F_i \right| u H. \quad (\text{A.0.57})$$

So, we have that

$$\partial_t N = -u H N + \sum_j \frac{(-1)^{j-1}}{\left| \bigwedge_i F_i \right|} * \left\{ [(\partial_j F_t)^j F_j + (\partial_j F_t)^N N] \wedge \bigwedge_{i \neq j} F_i \right\}, \quad (\text{A.0.58})$$

$$= -\left(u H - (\partial_i F_t)^i \right) N + \sum_j \frac{(-1)^{j-1} (\partial_j F_t)^N}{\left| \bigwedge_i F_i \right|} * \left(N \wedge \bigwedge_{i \neq j} F_i \right). \quad (\text{A.0.59})$$

Since $F_j \wedge (-1)^{j-1} \wedge N \wedge \bigwedge_{i \neq j} F_i = -N \wedge \bigwedge_i F_i$, we may compute in local geodesic coordinates that

$$\partial_t N = -\left(u H - (\partial_i F_t)^i \right) N - g^{ij} \langle \partial_i F_t, N \rangle F_j. \quad (\text{A.0.60})$$

Now, take note that

$$-g^{ij} \langle \partial_i F_t, N \rangle F_j = -g^{ij} (\partial_i u) F_j + g^{ij} \langle F_t, \partial_i N \rangle F_j, \quad (\text{A.0.61})$$

$$= -g^{ij} (\partial_i u) F_j. \quad (\text{A.0.62})$$

Also, observe that

$$u H - (\partial_i F_t)^i = 0. \quad (\text{A.0.63})$$

Combining (A.0.60), (A.0.62), and (A.0.63), we get (A.0.54).

□

Now, we prove Lemma 2.

Proof of Lemma 2. We may use (A.0.51) with $f = -\frac{|x|^2}{4}$ to get that

$$\frac{\partial}{\partial t} A_f(U_t) = \int_{U_t} (\langle \bar{\nabla} f, N \rangle u + H u) dA_\mu. \quad (\text{A.0.64})$$

So, using (A.0.39) we have that

$$\frac{\partial^2}{\partial t^2} A_\mu(U_t) \Big|_{t=0} = - \int_U u (\Delta u + |A|^2 u) dA_\mu + \int_U u \partial_t \langle \bar{\nabla} f, N \rangle dA_\mu \quad (\text{A.0.65})$$

$$+ \int_U (\langle \bar{\nabla} f, N \rangle + H) \partial_t (u dA_\mu). \quad (\text{A.0.66})$$

Now, for a critical hyper-surface, we have that $\langle \bar{\nabla} f, N \rangle + H = C$ a constant. Therefore, for a volume preserving variation, we have that

$$\int_U (\langle \bar{\nabla} f, N \rangle + H) \partial_t (u dA_\mu) = C \int_U \partial_t (u dA_\mu), \quad (\text{A.0.67})$$

$$= C \partial_t \left(\int_U u dA_\mu \right), \quad (\text{A.0.68})$$

$$= 0. \quad (\text{A.0.69})$$

From (A.0.54), we see that

$$\partial_t \langle \bar{\nabla} f, N \rangle = \overline{\text{Hess}_f}(N, N) u - \langle \bar{\nabla} f, \nabla u \rangle, \quad (\text{A.0.70})$$

$$= -\frac{1}{2} u + \frac{1}{2} \nabla_{x^T} u. \quad (\text{A.0.71})$$

Therefore, using (A.0.66), (A.0.69), and (A.0.71), we get that

$$\frac{\partial^2}{\partial t^2} A_\mu(U_t) \Big|_{t=0} = - \int_U u \left(\Delta u - \frac{1}{2} \nabla_{x^T} u + |A|^2 u + \frac{1}{2} u \right) dA_\mu \quad (\text{A.0.72})$$

□

Now, we prove a lemma relating bounds on the mean curvature of a hyper-surface to a mean-value inequality, as found in McGonagle-Ross[27]. The proof is based on the mean value inequality

for minimal surfaces [9].

Lemma 18. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hyper-surface with $|H| \leq M$. Also let $f \in C^2(\Sigma)$ satisfy $f \geq 0$ and $\Delta f \geq -\lambda t^{-2} f$ on $B_t(x) \subset B_R(0)$ for some λ . Then, for $s \leq t$ we have that*

$$\omega_n f(x) \leq e^{\left(\frac{\lambda}{2t} + M\right)s} s^{-n} \int_{B_s(x) \cap \Sigma} f dA. \quad (\text{A.0.73})$$

Here ω_n is the volume of the n -dimensional unit sphere in \mathbb{R}^n .

Proof. For general $\Sigma^n \subset \mathbb{R}^{n+1}$, we have that $\Delta|x|^2 = 2n - 2\langle x, N \rangle H$. By translation, we consider $\Sigma \subset B_s(0)$. So, we then have that

$$2n \int_{B_s \cap \Sigma} f dA = \int_{B_s \cap \Sigma} f \Delta|x|^2 dA + 2 \int_{B_s \cap \Sigma} f \langle x, N \rangle H dA, \quad (\text{A.0.74})$$

$$\begin{aligned} &= \int_{B_s \cap \Sigma} |x|^2 \Delta f dA + 2 \int_{\partial B_s \cap \Sigma} f |x^T| dA - s^2 \int_{B_s \cap \Sigma} \Delta f dA + 2 \int_{B_s \cap \Sigma} \langle x, N \rangle H f dA. \end{aligned} \quad (\text{A.0.75})$$

Now we let $g(s) = s^{-n} \int_{B_s \cap \Sigma} f dA$. Using the coarea formula and (A.0.75), we have that

$$\begin{aligned} g'(s) &= -ns^{-n-1} \int_{B_s \cap \Sigma} f dA + s^{-n} \int_{\partial B_s \cap \Sigma} f \frac{|x|}{|x^T|} dA, \\ &= -s^{-n-1} \left(\frac{1}{2} \int_{B_s \cap \Sigma} (|x|^2 - s^2) \Delta f dA + \int_{\partial B_s \cap \Sigma} f |x^T| dA + \int_{B_s \cap \Sigma} \langle x, N \rangle f H dA \right) \\ &\quad + s^{-n} \int_{\partial B_s \cap \Sigma} f \frac{|x|}{|x^T|} dA, \\ &\geq \frac{1}{2} s^{-n+1} \int_{B_s \cap \Sigma} \Delta f dA - s^{-n-1} \int_{B_s \cap \Sigma} \langle x, N \rangle f H dA. \end{aligned} \quad (\text{A.0.77})$$

Note that the final inequality uses that $f \geq 0$.

From the assumption that $\Delta f \geq -\lambda t^{-2} f$ on B_t and the bound $|H| \leq M$, we have for all $s \leq t$ that

$$g'(s) \geq -\frac{\lambda}{2} s^{1-n} \int_{B_s \cap \Sigma} f t^{-2} dA - M s^{-1-n} \int_{B_s \cap \Sigma} s f dA, \quad (\text{A.0.78})$$

$$\geq \left(-\frac{\lambda}{2t} + M \right) g(s). \quad (\text{A.0.79})$$

So, we get that

$$\frac{d}{ds} \left(g(s) e^{\left(\frac{\lambda}{2t} + M\right)s} \right) \geq 0. \quad (\text{A.0.80})$$

Integrating this we get that

$$\omega_n f(p) \leq e^{\left(\frac{\lambda}{2t} + M\right)s} s^{-n} \int_{B_s \cap \Sigma} f \, dA. \quad (\text{A.0.81})$$

□

Note that by taking $f = 1$ in (A.0.73), we have that if $p \in \Sigma$ and $|H| \leq M$ on $B_t(p) \cap \Sigma$, then for all $s \leq t$, we have that

$$A_E(B_s(p) \cap \Sigma) \geq \omega_n e^{-Ms} s^n, \quad (\text{A.0.82})$$

where ω_n is the volume of the standard unit ball in \mathbb{R}^n .

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Curriculum Vitae

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